Transition from Poynting Vector to Instantaneous Power

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Abstract—In recent studies on the power concept, Poynting Vector is employed to derive Instantaneous Power and an important tool for investigation of the physical meaning of Apparent Power in nonsinusoidal conditions. Hitherto Instantaneous Power was described as a postulate in the literature and not exactly derived for general conditions. In this study, Poynting Vector is used for the exact derivation of power in the electrical circuits and explanation of the physical meaning of Apparent Power components in nonsinusoidal conditions for more general conditions.

Index Terms—Instantaneous Power, Poynting Vector, Nonsinusoidal Conditions.

I. NOMENCLATURE

\( \varepsilon \): Dielectric Permittivity (F/m)

\( \mu \): Magnetic Permeability (H/m)

\( \sigma \): Conductivity (s/m)

\( \mathbf{\Phi} \): Poynting Vector (Watt/m²)

\( \mathbf{E} \): Electric Field Vector (V/m)

\( \mathbf{H} \): Magnetic Field Vector (A/m)

\( \mathbf{D} \): Displacement Field Vector (C/m²)

\( \mathbf{B} \): Magnetic Induction Vector (Weber/m²)

\( \mathbf{J} \): Convection Current Vector (A/m²)

\( V \): Voltage RMS value (V)

\( I \): Current RMS value (A)

\( \phi \): Phase difference between voltage and current

\( V_n \): RMS value of voltage’s \( n^{th} \) harmonic (V)

\( I_n \): RMS value of current’s \( n^{th} \) harmonic (A)

\( \alpha_n \): Phase angle of voltage’s \( n^{th} \) harmonic

\( \delta_n \): Phase angle of current’s \( n^{th} \) harmonic

II. INTRODUCTION

In recent discussions [1-5] on power definitions, Poynting Vector is used for interpretation of power theory in sinusoidal and nonsinusoidal conditions. Poynting Vector gives information about flow of power in electromagnetic theory and it is given in Equation 1

\[
\oint \mathbf{\Phi} \cdot d\mathbf{s} - \iint \mathbf{E} \cdot \mathbf{J} \, dv = \frac{\partial}{\partial t} \iint \left[ \frac{\varepsilon}{2} |\mathbf{E}|^2 + \frac{\mu}{2} |\mathbf{H}|^2 \right] \, dv + \sigma \iint |\mathbf{E}|^2 \, dv
\]

where Poynting Vector is defined as

\[
\mathbf{\Phi} = \mathbf{E} \times \mathbf{H}.
\]

In reference [1, 2], the suitability of Poynting Vector for investigation of power properties in electrical circuit is questioned. On the other hand, Instantaneous Power is obtained in [3-5] from Poynting Vector for special conductor shapes with the help of Transmission Line Theory; however, a general derivation for all conditions is not validated. In [3-5], the Active and Reactive parts of Instantaneous Power are obtained for sinusoidal conditions. The usage of Poynting Vector is generally limited to the left hand side of Equation 1 i.e. Equation 2 in which \( \mathbf{E} \) and \( \mathbf{H} \) are related to current and voltage for special conditions. In some studies in the Electromagnetic literature, the right hand side of Equation 1 is implemented for investigation of energy conservation requirement of solved problem for nonsinusoidal conditions [6]. On the other hand, Classical Power Theory in electrical engineering is based on Instantaneous Power, which has long been claimed as a postulate [7, 8] for AC circuits. Thus, the derivation of Instantaneous Power by means of Poynting Vector is an important work which theoretically explains the meaning of Classical Power concept, i.e. Active, Reactive and Apparent Powers.

In this paper, Instantaneous Power, which is the source of Classical Power concept, is exactly derived from the flux of Poynting Vector for a simple conductor, which is generally valid for all electrical circuits. In addition, the components of Instantaneous Power for nonsinusoidal conditions are discussed by means of this derivation to facilitate a physical explanation of power components.
III. REVIEW OF POWER THEORY IN SINUSOIDAL AND NONSINUSOIDAL CONDITIONS

Apparent Power can be derived from Instantaneous Power. For this aim, firstly, Instantaneous Power for sinusoidal conditions is expressed as

\[ p(t) = v(t) \cdot i(t) = 2 \cdot V \cdot I \cdot \sin(\omega t \cdot \sin(\omega t - \varphi)) \]  

(3)

with Instantaneous Voltage and Current, respectively;

\[ v(t) = \sqrt{2} \cdot V \cdot \sin(\omega t) \]  

(4)

\[ i(t) = \sqrt{2} \cdot I \cdot \sin(\omega t - \varphi). \]  

(5)

Secondly, Instantaneous Power can be divided into two orthogonal components as;

\[ p(t) = p_a(t) + p_r(t) \]  

(6)

with Instantaneous Active Power and Reactive Power, as given in Equation 7 and Equation 8.

\[ p_a(t) = V \cdot I \cdot \cos\varphi \]  

(7)

\[ p_r(t) = V \cdot I \cdot \sin\varphi \cdot \sin(2\omega t) . \]  

(8)

Finally, power components, in phasor domain, are defined as Active, Reactive and Apparent Powers:

Active Power is the average of Instantaneous Power,

\[ P = V \cdot I \cdot \cos\varphi. \]  

(9)

Reactive Power is the sup norm, which is the maximum value of a vector, of Instantaneous Reactive Power,

\[ Q = V \cdot I \cdot \sin\varphi. \]  

(10)

Apparent Power is defined as the vector sum of Active and Reactive Powers,

\[ S^2 = P^2 + Q^2. \]  

(11)

Up to now, one can see that Apparent Power is based on Instantaneous Power and it is claimed to be a postulate [7, 8].

In nonsinusoidal conditions, Instantaneous Power is identified as below

\[ p(t) = V_0 \cdot I_0 + V_0 \cdot \sum_{n \in N^*} \sqrt{2} \cdot I_n \cdot \sin(\omega t - \delta_n) + \sum_{n \in N^*} \sqrt{2} \cdot V_n \cdot \sin(\omega t - \alpha_n) \cdot I_0 + \sum_{n \in N^*} \sqrt{2} \cdot V_n \cdot \sin(\omega t - \alpha_n) \cdot \sum_{m \in N^*} \sqrt{2} \cdot I_m \cdot \sin(\omega t - \delta_m) \]  

(12)

where Instantaneous Voltage and Current can be written as

\[ v(t) = V_0 + \sum_{n \in N^*} \sqrt{2} \cdot V_n \cdot \sin(\omega t - \alpha_n) \]  

(13)

\[ i(t) = I_0 + \sum_{n \in N^*} \sqrt{2} \cdot I_n \cdot \sin(\omega t - \delta_n) \]  

(14)

where \( N^* \) is positive integers and \( I_0 \) and \( V_0 \) are the mean values of current and voltage. Therefore, the decomposition of Instantaneous Power in nonsinusoidal conditions is a difficult task due to the fact that cross products between the different harmonics of voltage and current exist. Thus, several different Apparent Power decompositions are proposed in the literature [9]. However, two decompositions of Apparent Power are generally accepted [10], namely; Budeanu’s and Fryze’s Apparent Power decompositions.

Budeanu postulated that total power is consist of two orthogonal components, namely Active and Deactive Powers [11]. Active Power could easily be calculated by averaging Instantaneous Power in time domain or by convolution in phasor domain,

\[ P = V_0 I_0 + \sum_{n \in N^*} V_n I_n \cdot \cos\varphi_n. \]  

(15)

where \( \varphi_n = \alpha_n - \delta_n. \)

This component was chosen by Budeanu due to the fact that it is actual power converted to the physical work. On other hand, Deactive Power, was divided into two components as Budeanu’s Reactive and Distortion Powers. Budeanu’s Reactive Power is calculated by summing of the individual harmonic Reactive Powers,

\[ Q_n = \sum_{n \in N^*} V_n I_n \cdot \sin\varphi_n. \]  

(16)

Distortion Power is calculated by the cross product of different harmonics voltages and currents,

\[ D_s = (S^2 - P^2 - Q^2)^{1/2} = \sum_{n \in N^*} V_n^2 I_n^2 + V_n^2 I_k^2 - 2V_n V_k I_n I_k \cdot \cos(\varphi_n - \varphi_k) \]  

(17)

Budeanu’s Reactive Power can completely be compensated with a simple capacitor. However, this is not possible in the case of Distortion Power given in Equation 17.

Despite the fact that Budeanu’s power decomposition provides information on completely compensable Reactive Power, the calculated Reactive Power does not provide any information about the source efficiency and optimum compensation capacitance. In addition, Budeanu’s decomposition require calculations in harmonic domain and sophisticated measurement devices. Fryze proposed a current based decomposition [12] in which current is divided into two orthogonal components namely; Active and Reactive Currents. The first is calculated by using Active Power

\[ i_a(t) = \frac{P}{V^2} v(t) \]  

(18)

and the second

\[ i_r(t) = i(t) - i_a(t). \]  

(19)

The power decomposition as suggested by Fryze is

\[ S^2 = P^2 + Q^r. \]  

(20)
with Active power
$$P = V I_a$$  \hspace{1cm} (21)$$

Fryze’s Reactive Power
$$Q_f = V I_r .$$  \hspace{1cm} (22)$$

The main advantages of Fryze’s decomposition are; to provide accurate information on source efficiency and to be determined by using ordinary phasor measurement devices. However, calculated values are not again suitable for Reactive Power Compensator design by using a simple capacitor. Therefore, controversy on decomposition of Apparent Power still goes on.

IV. DERIVATION OF INSTANTANEOUS POWER FROM POYNTING VECTOR

In this section, Instantaneous Power is exactly obtained by means of Poynting Vector. Consequently, the properties of Apparent Power components are investigated in the light of obtained Instantaneous Power’s parts.

Poynting Vector can be derived as given in Equation 1, which is Energy Conservation Equation, from Maxwell equations defined as

$$\text{rot}\vec{E} + \frac{\partial}{\partial t} \vec{B} = 0$$  \hspace{1cm} (23)$$

$$\text{rot}\vec{H} - \frac{\partial}{\partial t} \vec{D} = \vec{J}_v + \sigma \vec{E}$$  \hspace{1cm} (24)$$

The each part of Energy Conservation Equation has a special meaning i.e. the power transformed to heat energy and stored as electromagnetic energy in any medium.

The components of Energy Conservation Equation and their physical meanings are given in Table 1.

<table>
<thead>
<tr>
<th>Part</th>
<th>Physical Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_S \vec{E} \cdot d\vec{s}$</td>
<td>Power is passing through S surface by external sources</td>
</tr>
<tr>
<td>$\iiint_{\text{v}} \vec{E} \cdot \vec{J}_v , dv$</td>
<td>Power is consumed in S surface by sources in conductor.</td>
</tr>
<tr>
<td>$\sigma \iiint_{\text{v}}</td>
<td>\vec{E}</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial t} \iiint_{\text{v}} \left[ \frac{1}{2}</td>
<td>\vec{E}</td>
</tr>
</tbody>
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In order to facilitate the transformation from Poynting Vector to Instantaneous Power in electric circuits, a basic wire infinitely long model with finite radius and conductivity shown in Figure 1, can be used.

![Figure 1: A basic line model.](image)

It is shown in Figure 1 that electromagnetic energy flows through the S surface of the conductor. The main assumption is that z-axis components of Electric (\(\vec{E}\)) and Magnetic (\(\vec{H}\)) Fields are zero in the sense of transmission line approach. It means that Transverse Electromagnetic Magnetic Field propagation (TEM) is assumed along on infinitely long and infinitely thin wire and it is a perfect conductor (\(\sigma \rightarrow \infty\)). In addition, the wave length of the frequency for most power applications is much higher than the physical radius of the conductor. Due to the fact that any source does not exist in conductor, Equation 26 is zero and only Poynting Vector flux remains in the left hand side of Equation 1. Then, Instantaneous Active and Instantaneous Reactive Powers can be derived from Equation 27 and Equation 28.

Equation 27 gives information about power, which is converted to heat energy. This part is defined as Instantaneous Active Power in electrical circuits. That is derived from Equation 27 by the process which is detailed here:

Substituting the absolute value of Electric Field (\(\vec{E}\))

$$|\vec{E}| = \sqrt{E_x^2 + E_y^2}$$

in Equation 27, results in

$$P_a(t) = \sigma \iiint_{\text{v}} (E_x^2 + E_y^2) \, dv$$  \hspace{1cm} (30)$$

By means of the electrically short length assumption, the expression in Equation 30 can be written as

$$P_a(t) = \sigma \left( \int_x E_x \, dx \int_y E_y \, dy \right) dz + \sigma \left( \int_x E_x dx \int_y E_y dy \right) dz$$  \hspace{1cm} (31)$$

And then, if the x and y axis components of voltage and current, which are explained in Equation 32 and Equation 33, are inserted to Equation 31.
transition from Equation 31 to Equation 34 is established.

\[ p_x(t) = \sigma \int (v^2 + v^2_y) dy \]

Finally, integrating Equation 34 after substituting the absolute value of the voltage, which can be expressed as

\[ v^2 = v^2_x + v^2_y \]

and conductivity, which can be calculated as

\[ \sigma = \frac{1}{R \cdot z} \]

in Equation 34, Instantaneous Active Power is defined as

\[ p_x(t) = \frac{v^2(t)}{R} \]  

Equation 28 gives information on the power, which is stored as electromagnetic energy. This part can be evaluated as Reactive Power in an electrical circuit. Instantaneous Reactive Power can be derived by substituting Electric Field \((E_x)\), given in Equation 29, and Magnetic Field \((H_x)\)

\[ \|\vec{H}\| = \sqrt{H^2_x + H^2_y} \]

in Equation 28, results in

\[ p_r(t) = \frac{\partial}{\partial t} \iint_S \left[ \frac{\sigma}{2} \left( E_x^2 + E_y^2 \right) + \frac{\mu}{2} \left( H_x^2 + H_y^2 \right) \right] dy \]

And then, by means of electrically short length assumption, the expression in Equation 39 can be written as

\[ p_r(t) = \frac{\partial}{\partial t} \iint_S \left[ \frac{\sigma}{2} \left( \int E_x \, dx \int E_y \, dy \right) \right] \, dz + \frac{\mu}{2} \iint_S \left[ \int H_x \, dx \int H_y \, dy \right] \, dz \]

By inserting x and y axis components of voltage and current, which are defined as Equation 32 and Equation 33, again transition from Equation 40 to Equation 41 is established.

\[ p_r(t) = \frac{\partial}{\partial t} \int (v_x^2 + v_y^2) \, dz \]

By integrating both terms of Equation 41 and voltage \((v)\), current \((i)\), capacitance \((C)\) and inductance \((L)\), which are defined as in Equation 35 and Equations (42-44), are considered

\[ i^2 = i_x^2 + i_y^2 \]

the expression in Equation 41 can be written as

\[ p_r(t) = \frac{\partial}{\partial t} \left[ \frac{C}{2} v_x^2 + \frac{L}{2} i_x^2 \right] \]

Equation 45 is transformed to Equation 46 by taking the partial derivative according to the fact that x and y axes are electrically short lengths

\[ p_r(t) = \frac{C}{2} \frac{dv_x}{dt} \frac{L}{2} \frac{di}{dt} \]

and finally, by inserting the relations given in Equation 47 and Equation 48 to Equation 46

\[ i_c = \frac{C}{L} \frac{dv_c}{dt} \]

Instantaneous Reactive Power in the time domain is expressed as

\[ p_r(t) = v_c(t) \cdot i_c(t) + v_c(t) \cdot i_c(t) \]

It is shown that total Instantaneous Power is written as the sum of Instantaneous Active and Reactive Powers:

\[ p(t) = v_x(t) \cdot i_x(t) + v_x(t) \cdot i_x(t) + v_x(t) \cdot i_x(t) \]

As a result, according to the fact that any electrical circuit can be converted to a serial or parallel combination of RLC element, Equation 50 is transformed into Equation 51 or Equation 52

\[ p(t) = v_x(t) \cdot i_x(t) + v_x(t) \cdot i_x(t) + v_x(t) \cdot i_x(t) \]

Thus, Instantaneous Power concept is exactly derived from Poynting Vector for an infinitely long wire having finite radius and finite conductivity.

V. CONCLUSION

Hitherto Instantaneous Power, \(p(t)=v(t) \cdot i(t)\), is claimed as a postulate in the literature. Although there are some attempts to derive the Instantaneous Power from Poynting Vector, it is not clearly derived for general conditions and using right hand side of Energy Conservation Equation in accessible literature. In this work, Poynting Vector is used to extract Instantaneous Power without any special assumptions of conductor’s shape and circuits.

It is shown that Poynting Vector could be used to find the physically interpretation for the Instantaneous Power, which is the source of Apparent Power. The transition between Poynting Vector and Instantaneous Power is exactly obtained for an infinitely long wire having finite radius and conductivity. Furthermore, the transition is valid and identical for Sinusoidal and Nonsinusoidal conditions. Equation 37 and 49 shows that Instantaneous
Power is consist of only Active and Reactive parts for all conditions.

It is known that Transverse Electromagnetic Magnetic Field propagation (TEM), used in this paper, which is based on infinitely long, infinitely thin and perfect conductor assumption and it is questioned in the Electromagnetic literature [13]. Accordingly, Transverse Electromagnetic Field (TE) and Transverse Magnetic Field (TM) modes should further be investigated on the sense of nonsinusoidal excitation. That will be considered as a future work.

VI. REFERENCES


