1 INTRODUCTION

Investigation of underwater acoustic problems in a time domain is a crucial topic, especially for wide-band acoustic sources such as short sonar pulses, air-guns and underwater explosions. Principally, analytical time domain solutions can be obtainable by two fundamental ways: First, the time domain responses (transient) from frequency domain solutions can be extracted by the transformation methods (such as Fourier transformation etc.). Second, a direct analytical time domain solution can be handled. In the solution of underwater acoustic problems, Normal Mode, Parabolic Equation and Ray methods are widely used. However, Normal Mode method is a more critical one because it gives a full wave solution and valid for all frequencies. The Parabolic Equation method is based on the paraxial approximation and gives a one-way wave solution. The Ray method gives the approximate solution of a wave equation especially for high frequencies with disadvantages such as caustic problems, weakness in a cut-off analysis, etc. Therefore, frequency domain Normal Mode method is used especially for underwater passive acoustic problems. However, its interpretation for time domain responses of the wide-band acoustic signal is problematic in the frequency domain. Therefore, a direct analytical Time Domain Normal Mode (TDNM) method is necessary for the practical applications. In this work, an application of the TDNM method for an isovelocity waveguide with pressure release walls is shown with the validations of an analytical Time Domain Image (TDI) method and a numerical Finite Difference Time Domain (FDTD) method. The weak points of them against the TDNM method are shown. The importance of the TDNM solutions is discussed in details.

2 NEED FOR TIME DOMAIN SOLUTIONS

One of the widely used ways for obtaining time domain responses in the underwater acoustic problems is to apply the inverse Fourier transformation through the frequency domain solutions. However, the classical Fourier analysis has the following disadvantages:

- It needs multiple runs of a frequency domain model in a set of discrete frequencies. However, it is a troublesome way because the frequency responses at many frequencies must be independently calculated with a high degree of precision for desired frequency resolution. This leads to intensive computational calculations and long computational times.
- The excitations cannot be Fourier transformable. This critically limits the applicability of this technique.
- The effect of initial conditions cannot be simply handled in the Fourier transformed solutions because it is a double side transform having an unphysical part tends to negative times (negative frequencies). This leads to violate the causality principle in the solution.
- In the presence of nonlinearity, interactions among the frequency components invalidate the frequency domain transformed time-domain approach [1].

All these reasons can cause inaccurate calculations, especially for the late time part of the transients. Thus, this technique will be questionable. These disadvantages are also valid in obtaining the time domain solution from the frequency domain based Normal Mode solutions. Alternatively, at a first glance, the numerical time domain methods such as Finite Difference Time
Domain (FDTD) solutions can be preferable because it gives a full wave solution, including the initial conditions leads to satisfy the causality principle. However, the FDTD solutions have inherent weakness such as numerical dispersion effect disperses the solution numerically and leads to the inaccurate results. Moreover, the FDTD method needs unacceptable calculation times and computational power, especially for large-scale problems. It is well known that the acoustic problems are generally in the classes of the large-scale problems. Therefore, the application of the direct analytical Time Domain Normal Mode (TDNM) method is unavoidable for the wide-band classes of the underwater acoustical problems. Thus, the authors are focused on the direct, analytical and causal TDNM solutions. A benchmarking problem of an isovelocity acoustic waveguide with the pressure release boundaries is considered in order to show the accuracy of the TDNM method comparing with the Time Domain Image (TDI) and the FDTD methods.

3 TIME DOMAIN NORMAL MODE METHOD

The frequency domain Normal Mode method is widely used especially for the passive underwater acoustic problems. It is based on a solution of a Helmholtz equation by using separation of variables technique. However, this method is practical for narrow-band signals. In the case of the wide-band signals such as different acoustic pulses, the Time Domain Normal Mode (TDNM) method is needed due to the above-mentioned problematic reasons. It is also known that the propagation of acoustic pulses, which permits detailed results cannot be described by a simple analysis in the frequency domain [2].

Leaving aside the inverse Fourier transformed based solutions and some efforts in electromagnetics [3]-[4], it seems that Anderson and Barnes first showed a fundamental formulation to obtain the direct analytical time domain Normal Mode solutions [5]. However, they did not solve any problem and focused only to experimental investigations. Then, Coppens used the same idea for investigation of acoustic transients [6]. These works were based on the Laplace transform solution of the homogeneous (source-free) time domain Klein Gordon equation without source term. However, the source term representing the location and time dependency of radiated noise sources of ships, underwater explosions, submarines, air-guns etc. is very crucial in underwater acoustic problems. Therefore, it is necessary to investigate the solution of the inhomogeneous (source-present) wave equation in the acoustic channel. Furthermore, the source induced pulse time spreading should be taken into account for accurate calculation of Sound Pressure Level (SPL).

The proposed TDNM method gives a direct analytical and causal full wave mode solution [7]. It is based on the incomplete separation of variables technique applied for the inhomogeneous time domain acoustic wave equation in two-dimensional Cartesian coordinates as

\[
\frac{\partial^2 u(x, z, t)}{\partial x^2} + \frac{\partial^2 u(x, z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 u(x, z, t)}{\partial t^2} = -\delta(x)\delta(z - z_o)\tau(t) \tag{1}
\]

where \( u(x, z, t) \) is the acoustic pressure, \( c \) is the sound speed, \( z_o \) is the source depth and \( \tau(t) \) is the arbitrary time dependency of the acoustic sources. After separation, an ordinary Sturm-Liouville type and a partial Klein-Gordon differential type equations are obtained. The Sturm-Liouville type differential equation can be solved in a classical manner. The solution of the inhomogeneous Klein-Gordon equation is found by a time domain Green Function technique. The time domain Green function of the Klein-Gordon equation \( G = G_n \) for mode solutions is [11]

\[
G_n = \frac{c}{2} J_0 \left( \chi_{xtn} \sqrt{c^2(t - t')^2 - (x - x')^2} \right) H(t - t') - |x - x'| \tag{2}
\]

where \( \chi_{xtn} \) is the mode eigenvalues of the Klein-Gordon Equation, \( J_0(\cdot) \) is the zero-order Bessel function of the first-kind and \( H(\cdot) \) is the Heaviside-Step function. The initial conditions are
Then, the general solution of \( u(x, z, t) \) can be constructed as below:

\[
\begin{align*}
  u(x, z, t) &= \sqrt{\frac{2\rho_0}{h}} \sum_{n=1}^{\infty} a_n s \sin \left( \frac{n\pi z}{h} \right) \left[ \int_{0}^{\infty} f(x', t') G_n dx' dt' - \frac{1}{c^2} \int_{-\infty}^{0} \left( \frac{\partial G_n}{\partial x'} - \frac{G_n}{h} \right) dx' \right] \\
  \text{where } f(x', t') \text{ is the source function and } G_n = G_n(x, t|x', 0) \text{ [7]. This expression gives the acoustic pressure in the time domain. The integrals in the expression can be calculated analytically or numerically depending on the form of the source function.}
\end{align*}
\]

A single layer acoustic (ideal) waveguide with the pressure release sea surface and bottom models is considered as a numerical example. The sound speed is assumed to be a constant, such as in a Pekeris waveguide, is also feasible in the Normal Mode method because of not strong influence on the cut-off frequencies, especially in the slowly varying profiles. Furthermore, the chosen model can be used for some classes of shallow water acoustic problems with strong bottom reflections [8].

The extension of the TDNM method is also planned as a future work for more realistic problems such as the Pekeris waveguide with lossless and lossy bottoms. For this aim, the lower layer of the acoustic waveguide can be replaced with the equivalent pressure-release surface depth by using the effective depth concept. Thereby, the modified ideal waveguide can be used to calculate the acoustic fields in the more realistic mediums by the “effective” pressure-release bottom lying below the actual bottom interface [9], [10].

## 4 BENCHMARKS

The benchmarkings of the TDNM results are performed by the numerical FDTD method and analytical Time Domain Image (TDI) method. The causal cosine, causal Gaussian pulse, unweighted and weighted cosine pulse types sources are used in the comparison between the TDNM, the FDTD and TDI methods.

### 4.1 Benchmarking with the Finite Difference Time Domain Method

The FDTD is a numerical method which gives a direct time domain solution. It can be used as a validation tool for the underwater acoustic problems [12]. The FDTD method has disadvantages for the large-scale problems. In this case, the numerical Pseudo Spectral Time Domain method can be preferable.

**First**, the FDTD results are obtained for the causal cosine source \( f(x, t) = \delta(x) \cos(\omega t) H(t) \) with the homogenous initial conditions. The first-order Mur type boundary condition is used to terminate the problem space. The parameters of the problem; operational frequency, \( f = 25 \text{ Hz} \), depth of the channel, \( h = 50 \text{ m} \), sound speed, \( c = 1500 \text{ m/s} \), range \( x = 1 \text{ km} \) and source and observations points, \( z_s = z_r = 25 \text{ m} \). The accuracy of the FDTD solution is dependent on the spatial and temporal grid sizes. The stability criterion and numerical dispersion errors are the most limiting points in the FDTD method. Therefore, The TDNM result is compared with the FDTD results for different spatial step sizes, \( \Delta x = \lambda/60 \), \( \Delta x = \lambda/120 \) and \( \Delta x = \lambda/240 \) (\( \Delta x = \Delta z \)). The integral in the TDNM solution for the steady-state regime can be calculated, analytically [13]. The transient part of them can be numerically calculated by an adaptive Gauss-Kronrod quadrature method [13]. The compared TDNM results are shown in Figure 1.
Figure 1. Comparison between the TDNM and the FDTD results for the causal cosine source.

It is seen that a good agreement is obtained especially for smaller spatial step sizes of the FDTD method. However, the smaller spatial time steps lead to large computational loads in the FDTD solution. The TDNM solution is an analytical solution that does not have this type of restriction.

Second and third comparisons between the TDNM and FDTD methods are shown for a causal Gaussian pulse, \( f(x,t) = \delta(x) e^{-((t-t_0)/\beta)^2} H(t) \) where \( \beta = 0.003 \) and \( t_0 = 0.03 \) s and an unweighted cosine pulse with the frequency is 150 Hz and the pulse length is 0.05 s. They are shown in Figure 2 and 3, respectively. The maximum mode number for the TDNM solution is found by frequency spectrum of these pulses.

The compared TDNM result of the causal Gaussian pulse for three different FDTD spatial step size \( \Delta x = \lambda/10 \), \( \Delta x = \lambda/30 \) and \( \Delta x = \lambda/60 \) are shown in Figure 4.
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Figure 4. Comparison between the TDNM and the FDTD results for the causal Gaussian pulse.

The smaller step size such as $\Delta x = \lambda/120$ or $\Delta x = \lambda/240$ are not practical because their FDTD calculations need very long computational times. Here, the wavelength should be chosen for the highest frequency component of the causal Gaussian pulse. Therefore, the unreliable results may be obtained by using the inconvenient spatial and temporal grids. On the other hand, the TDNM solution may need more computational power for the numerical evaluation of the integral. However, this is not as much as the FDTD needs.

The compared TDNM result of the unweighted cosine pulse for three different FDTD spatial step sizes of $\Delta x = \lambda/30$, $\Delta x = \lambda/60$ and $\Delta x = \lambda/120$ are shown in Figure 5.

Figure 5. Comparison between the TDNM and the FDTD results for the unweighted cosine pulse.

4.2 Benchmarking with the Time Domain Image Method

The Time Domain Image (TDI) method is also a valuable analytical tool for the time domain comparisons. It can be accurately used in the case of the pressure release boundaries (the ideal waveguide). The TDI method is based on the usage of the time domain Green’s function of the wave equation. Two-dimensional time domain Green’s function of the wave equation is
\[ G(x, z, t) = \frac{1}{2\pi} \frac{H(t - x/c)}{\sqrt{t^2 - (x/c)^2}} \]  

(5)

Then, the acoustic pressure can be calculated with a convolution integral, \( \otimes \) as

\[ u(x, z, t) = \sum_{m=0}^{\infty} T(t) \otimes [G(R_{m1}, z, t) - G(R_{m2}, z, t) - G(R_{m3}, z, t) + G(R_{m4}, z, t)] \]  

(6)

where \( T(t) \) is the source function, \( R_{mn} = \sqrt{x^2 + z_{mn}^2} \) is the distance between the image and observation points, \( z_{mn} \) is the depth distance for every images. \( m \) is the number of image (range dependent). \( n = 4 \) and shows the chosen propagation paths as direct, bottom bounce, surface bounce, surface and bottom bounces. These convolution integrals are calculated numerically. Particularly, the comparisons between the TDNM and the TDI methods are shown for the same casual Gaussian pulse, and unweighted cosine pulses in Figure 6 and Figure 7, respectively.

![Figure 6. Comparison between the TDNM and the TDI results for the causal Gaussian pulse.](image)

![Figure 7. Comparison between the TDNM and the TDI results for the unweighted cosine pulse.](image)
The good agreement is obtained between the TDNM and the TDI results. This result also emphasizes the effect of the FDTD errors over the results shown in Figure 1 and Figure 3.

Lastly, the comparisons between the TDNM and the TDI methods are shown for a causal Gaussian weighted cosine pulse, \( f(x, t) = \delta(x) e^{-(t-t_0)/\beta^2} \cos(\omega t)H(t) \) where \( \beta = 0.005 \), \( t_0 = 0.25 \) s and \( f = 100 \) Hz. It is shown in Figure 8.

![Figure 8. A Gaussian weighted cosine pulse.](image)

In Figure 9, the compared results between the TDNM and the TDI methods are shown for the propagated Gaussian weighted cosine pulse.

![Figure 9. Comparison between the TDNM and TDI results for the Gaussian weighted cosine pulse.](image)

The good agreement is observed between the compared results. It leads the accuracy of the proposed method.

5 CONCLUSIONS

The time domain solution of the inhomogeneous wave equation for different types of transient acoustic sources such as underwater explosions, air-guns is investigated by the analytical Time Domain Normal Mode method. Particularly, the causal cosine, causal Gaussian pulse, unweighted and weighted cosine pulse types sources are used to excite the underwater acoustic waves. The obtained results for an ideal waveguide are compared with the FDTD and the TDI results in the time
domain. It is shown that the FDTD results need smaller step sizes for acceptable accurate results. This needs more computational power and computational sources. Therefore, the FDTD method may not be a practical solution, especially for large-scale underwater acoustic problems. On the other hand, the TDI results are in the good agreement with the TDNM results. However, the TDI results need to calculate more than one numerical convolution integral. Nevertheless, the TDNM method gives the same result with less effort because the convolution integral can be analytically taken in the TDNM method for some cases. The computational cost for the TDNM method comes from the numerical solution of the time integral for some types of the sources. As future works, the applications of the TDNM method for the problems of the Pekeris waveguide, time dependent rough sea surfaces and lossy sea beds are planned by the authors.

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REFERENCES