STUDY OF A TIME VARIANT CAVITY SYSTEM

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Abstract—A cavity with a time signal applied taken jointly is treated as a time variant physical system. Standard formulation of the boundary-value problem for system of Maxwell's equations (with \( \partial_t \)) is supplemented with the initial conditions and the causality principle. It is proved that electromagnetic field can be presented as the classical decompositions in terms of the solenoidal and irrotational modes but with time dependent modal amplitudes. For the latters, evolutionary (i.e., with time derivative) ordinary differential equations are derived and solved analytically, in quadratures. Simple explicit solutions (in elementary functions) for the modal amplitudes as functions of time are obtained in a particular case. Numerical examples are exhibited, time-domain resonances are studied.

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References

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1. INTRODUCTION

To begin with, let us imagine schematically a standard cavity (cylindrical, for example) as an element of some practical microwave system. When configured-in any microwave scheme, the cavity must include an input item for a signal supplied and some output for further signal processing.

We select now the cavity with a given input signal supplied jointly as the object for our study. As a model of the input signal, we introduce given densities of electric \( J_e (r, t) \) and/or magnetic \( J_m (r, t) \) currents with \( r \in V \), where \( V \) is volume of the cavity. Presence of position vector \( r \) at the argument \((r, t)\) is provided for description of the input unit position within \( V \); time \( t \) is intended for specifying the signal time-domain pattern. Specifically, every input signal in practice has a beginning and end. We assume therefore that

\[
J_e (r, t) \neq 0 \quad \text{and/or} \quad J_m (r, t) \neq 0, \quad \text{while} \quad 0 \leq t \leq T, \quad (1a)
\]

and identical zero otherwise. As regards the time dependence of the signal within \( 0 \leq t \leq T \), no other suppositions are imposed only that it is known.

So, we consider the cavity jointly with specified in such a way input signal as a time variant physical system [1]. Analysis will be performed with using a new time-domain method, which is named as Evolutionary Approach to Electromagnetics (EAE). Its methodological details will be explained further.

1.1. Formulation of the Problem

Now we proceed with exact formulation of the problem described above verbally. Of course, we should solve Maxwell’s equations within \( V \)

\[
\begin{align*}
\text{rot} \, H (r, t) & = \varepsilon_0 \varepsilon \, \partial_t E (r, t) + \sigma E (r, t) + J_e (r, t), \\
- \text{rot} \, E (r, t) & = \mu_0 \mu \, \partial_t H (r, t) + J_m (r, t),
\end{align*}
\]

(1b)

where the constitutive relations are used as

\[
D = \varepsilon_0 \varepsilon \, E, \quad B = \mu_0 \mu \, H, \quad J = \sigma E ; \quad (2)
\]

\( \varepsilon_0, \mu_0 \) are the free-space constants. Eqs. (2) means that the cavity is filled with a linear homogeneous lossy time-invariant medium. Electromagnetic parameters of the medium \( \varepsilon, \mu \) and \( \sigma \) are real constants. The system of differential Maxwell’s equations involves the div-equations as well. We omit them: they can be derived from Eqs. (1b) and the continuity equation when needed.
Differential Maxwell's equations (1b) are valid within open volume \( V \) (i.e., for \( r \in V, \ r \notin S \)). When surface \( S \) is supposed as perfectly conducting, Maxwell's equations take the algebraic forms of the boundary conditions over \( S \), namely:

\[
[n \times E] |_S = 0, \quad (n \cdot H) |_S = 0,
\]

where \( n \) is the outward unit vector normal to \( S \).

Maxwell's differential equations (1b) are evolutionary: i.e., they involve time derivative \( \partial_t \). Hence, initial conditions should be supplemented as

\[
E(r, t) |_{t=0} = E(r, 0), \quad H(r, t) |_{t=0} = H(r, 0),
\]

where \( E(r, 0), \ H(r, 0) \) should be given in closed domain \( r \in V, \ r \in S \).

In addition to mathematical formulation (1a)–(4) of the problem under study, we should impose two physical restrictions upon its solution sought for. First one is the requirement that energy of electromagnetic field must be finite, which is usually postulated as follows

\[
\int_{t_1}^{t_2} dt \int_V \left( \varepsilon_0 \varepsilon E \cdot E^* + \mu_0 \mu H \cdot H^* \right) dv < \infty,
\]

where \( 0 \leq t_1 < t_2 < \infty, \ V' \subseteq V \), and the star means complex conjugation. The latter means that the solution is sought for as complex valued vector functions \( E(r, t), \ H(r, t) \). The second one is the principle of causality

\[
E(r, t) = 0, \quad H(r, t) = 0, \quad \text{while} \ t < 0,
\]

which means that electromagnetic field, excited by the sources \( J_e(r, t), J_m(r, t) \), must be identical zero before they start to excite the cavity.

1.2. Discussion of Approaches to the Problem

Classical Method of Complex Amplitudes (MCA) operates mostly with time harmonic electromagnetic fields as

\[
E(r, t) = E(r, \omega) e^{-i\omega t}, \quad H(r, t) = H(r, \omega) e^{-i\omega t},
\]

where \( \omega \) is the frequency parameter, \( -\infty < \omega < \infty \), and vectors \( E, H \) are just the complex amplitudes of field \( E, H \). Within the frames of classical theory of cavities, the following fundamental fact has been established\(^\S\). Complex amplitudes \( E, H \) of arbitrary time harmonic

\(^\S\) Tracing literature back, one can disclose roots of very important idea that was put forward in 40s and 50s. Slater [2], Müller [3], Kisun'ko [4], and Kurokawa [5] were
field can be presented in cavities in the form of so-called eigenmodal decompositions\(^\dagger\) as

\[
E = \sum_{n=1}^{\infty} e'_n(\omega) E'_n(r) + \sum_{n=1}^{\infty} e''_n(\omega) E''_n(r) + \sum_{\alpha=1}^{\infty} a_\alpha(\omega) \nabla \Phi_\alpha(r),
\]

\[
H = \sum_{n=1}^{\infty} h'_n(\omega) H'_n(r) + \sum_{n=1}^{\infty} h''_n(\omega) H''_n(r) + \sum_{\beta=1}^{\infty} b_\beta(\omega) \nabla \Psi_\beta(r).
\]

(8)

The primed vectors at the series are specified in the general case via the following vector boundary eigenvalue problems for Laplacian:

\[
\begin{align*}
(\Delta + \omega'^2_n \varepsilon_0 \mu_0 \varepsilon \mu) E'_n &= 0, \quad \text{div } E'_n = 0, \quad [n \times E'_n] |_{S} = 0, \\
H'_n &= -i \text{rot } E'_n / \omega'_n \mu_0 \mu; \quad n = 1, 2, \ldots; \\
(\Delta + \omega''^2_n \varepsilon_0 \mu_0 \varepsilon \mu) H''_n &= 0, \quad \text{div } H''_n = 0, \quad (n \cdot H''_n) |_{S} = 0,
\end{align*}
\]

(9)

\[
\begin{align*}
E''_n &= i \text{rot } H''_n / \omega''_n \varepsilon_0 \varepsilon; \quad n = 1, 2, \ldots.
\end{align*}
\]

(10)

Since we have homogeneous equations at the first rows in Eqs. (9) and (10), vectors \(E'_n\) and \(E''_n\) can be defined as real functions of coordinates. Then vectors \(H'_n\) and \(H''_n\) should be pure imaginary ones.

Physically, solutions of (9) and (10) present \(TE\)- and \(TM\)-modes in any cylindrical cavity of arbitrary cross section that is filled with \textit{loss-free} medium specified by real parameters \(\varepsilon\) and \(\mu\). Quantities \(\omega'_n\) and \(\omega''_n\) are eigenfrequencies of appropriate modes. The series with these modes present in Eqs. (8) \textit{solenoidal} part\(^\dagger\) of total field \(E, H\). The series with potentials \(\Phi_\alpha\) and \(\Psi_\beta\) present decomposition of its \textit{irrotational}\(^+\) part. These potentials are specified via well-studied scalar Dirichlet and Neumann boundary eigenvalue problems for Laplacian as

\[
(\Delta + \kappa'^2_\alpha) \Phi_\alpha = 0, \quad \Phi_\alpha |_{S} = 0,
\]

(11)

\[
(\Delta + \nu^2_\beta) \Psi_\beta = 0, \quad (n \cdot \nabla \Psi_\beta) |_{S} = 0.
\]

(12)

\(^\dagger\) Decompositions (8) are valid for cavities with arbitrary singly connected (geometrically) surface \(S\) but perfectly conducting physically.

\(^\ddagger\) Since \(\text{div } E'_n = 0, \text{div } H'_n = 0, \text{div } E''_n = 0, \text{div } H''_n = 0\).

\(^+\) Since \(\text{rot } \nabla \Phi_\alpha = 0, \text{rot } \nabla \Psi_\beta = 0\).
When one operates within the frames of \( MCA \), only one way is free to solving the problem under consideration (1a)–(6). One can present its solutions as continual superposition of the time harmonic fields using Fourier transform as

\[
\mathcal{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) \, e^{-i\omega t} d\omega, \quad \mathcal{H}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathbf{H}(\mathbf{r}, \omega) \, e^{-i\omega t} d\omega.
\]  

(13)

However, physical intuition says that this way is questionable in the case of \textit{time-variant} physical systems. Indeed, solution sought for must satisfy the \textit{causality principle} (6). But time harmonic field (7) is steady-state and can exist within infinite time interval \(-\infty < t < \infty\) only. Form (13), as a continual superposition of \textit{stationary} fields (7), is natural for description of processes in \textit{time-invariant} systems. Studying \textit{time-variant} systems, however, needs another way.

Let us first present crucial ideas of our time-domain approach to the problem under study. Implementation them will be given in next sections in more detail.

Electromagnetic field \( \mathcal{E}(\mathbf{r}, t), \mathcal{H}(\mathbf{r}, t) \) sought for we are going to present in the same form of the eigenmodal decompositions (8) \textbf{but} with \textit{time} dependent modal amplitudes (instead of the \textit{frequency} dependent there): i.e., as

\[
\mathcal{E}(\mathbf{r}, t) = \sum_{n=1}^{\infty} e'_n(t) \mathbf{E}'_n(\mathbf{r}) + \sum_{n=1}^{\infty} e''_n(t) \mathbf{E}''_n(\mathbf{r}) + \sum_{\alpha=1} a_\alpha(t) \nabla \Phi_\alpha(\mathbf{r}),
\]

\[
\mathcal{H}(\mathbf{r}, t) = \sum_{n=1}^{\infty} h'_n(t) \mathbf{H}'_n(\mathbf{r}) + \sum_{n=1}^{\infty} h''_n(t) \mathbf{H}''_n(\mathbf{r}) + \sum_{\beta=1} b_\beta(t) \nabla \Psi_\beta(\mathbf{r}).
\]

(14)

Then we must obtain a problem for the modal amplitudes. In order to do this, we must derive the modal decompositions (14) from Maxwell's equations \textbf{with keeping} time derivative \( \partial_t \) there. It yields after some \textit{evolutionary} differential equations (i.e., with \textit{time derivative}) for the modal amplitudes.

This idea was proposed originally in 80s and presented in Russian journals; its English version was published in [7]. We name this method as \textit{Evolutionary Approach to Electromagnetics (EAE)} because the evolutionary differential equations are ultimately determinative in studying various time domain phenomena. An impression about abilities of the \textit{EAE} for development of Electromagnetics in the Time Domain one can get from fresh as yet previous investigations [8–13].
2. DERIVATION OF THE MODAL DECOMPOSITIONS

In this section, we'll derive and justify Eqs. (14) relying on Eqs. (8) as the fact established and proved in classical Electromagnetics.

In Eqs. (1b), let's multiply first row by $i / \varepsilon_0 \varepsilon$, the second one by $i / \mu_0 \mu$, where $i = \sqrt{-1}$, and write out them both as one 6-dimensional vector equation

$$
\left( \begin{array}{c}
(i / \varepsilon_0 \varepsilon) \rot \mathcal{H} \\
(-(i / \mu_0 \mu) \rot \mathcal{E}
\end{array} \right) =
\left( \begin{array}{cc}
\partial_t \mathcal{E} + (\sigma / \varepsilon_0 \varepsilon) \mathcal{E} + \mathcal{J}_e / \varepsilon_0 \varepsilon \\
\partial_t \mathcal{H} + \mathcal{J}_m / \mu_0 \mu
\end{array} \right).
$$

(15)

We can write out, in turn, left-hand side of Eq. (15) as follows

$$
\left( \begin{array}{c}
(i / \varepsilon_0 \varepsilon) \rot \mathcal{H} \\
(-(i / \mu_0 \mu) \rot \mathcal{E}
\end{array} \right) =
\left( \begin{array}{cc}
o & (i / \varepsilon_0 \varepsilon) \rot \\
-(i / \mu_0 \mu) \rot & o
\end{array} \right)
\left( \begin{array}{c}
\mathcal{E} \\
\mathcal{H}
\end{array} \right),
$$

where $o$ is $3 \times 3$ zero-valued matrix. Eq. (16) suggests to introduce $6 \times 6$ matrix differential procedure $\mathcal{R}'$ and 6-dimensional vector-column $\mathcal{X}$ as follows

$$
\mathcal{R} = \left( \begin{array}{cc}
o & (i / \varepsilon_0 \varepsilon) \rot \\
-(i / \mu_0 \mu) \rot & o
\end{array} \right), \quad \mathcal{X}(r, t) = \left( \begin{array}{c}
\mathcal{E}(r, t) \\
\mathcal{H}(r, t)
\end{array} \right).
$$

(17)

It is clear that in operation $\mathcal{R}' \mathcal{X}(r, t)$ procedure $\mathcal{R}'$ acts on space variables $r$ at the argument $(r, t)$, but time $t$ is a parameter in regard to $\mathcal{R}'$.

Now we joint differential procedure $\mathcal{R}'$ with the boundary conditions (3). It results in a bounded operator $\mathcal{R}$ defined as follows

$$
\mathcal{R} \mathcal{X}(r, t) = \left\{ \begin{array}{l}
\mathcal{R}' \mathcal{X}(r, t) \quad \text{while} \quad r \in V, \quad r \notin S; \\
[n \times \mathcal{E}]|_S = 0, \quad (n \cdot \mathcal{H})|_S = 0, \quad \text{while} \quad r \in S.
\end{array} \right.
$$

(18)

Then set of equations (1b), (3) can be written out as one operator equation

$$
\mathcal{R} \mathcal{X}(r, t) = i (\partial_t \mathcal{X}(r, t) + \mathcal{F}(r, t)),
$$

(19)

where $\mathcal{F}$ is 6-dimensional vector $col (\mathcal{J}_e / \varepsilon_0 \varepsilon, \mathcal{J}_m / \mu_0 \mu)$; $col$ means column.

Let's introduce a space of 6-dimensional vector functions of coordinates $\mathcal{X}(r)$ $(r \in V, r \in S)$, each of which satisfies the same boundary conditions as (3)

$$
\mathcal{X}(r) = \left( \begin{array}{c}
\mathbf{E}(r) \\
\mathbf{H}(r)
\end{array} \right), \quad [n \times \mathbf{E}(r)]|_S = 0, \quad (n \cdot \mathbf{H}(r))|_S = 0.
$$

(20)
Volumetric integration in Eq. (5) suggests to specify this space via inner product

\[(\mathcal{X}_1, \mathcal{X}_2) = \frac{1}{V} \int_V (\varepsilon_0 \varepsilon \mathbf{E}_1 \cdot \mathbf{E}_2^* + \mu_0 \mu \mathbf{H}_1 \cdot \mathbf{H}_2^*) \, dv,\]  

(21)

where \(\mathcal{X}_1 = \text{col} (\mathbf{E}_1, \mathbf{H}_1), \mathcal{X}_2 = \text{col} (\mathbf{E}_2, \mathbf{H}_2)\) are arbitrary pair of the vectors from this space. It is evidently that we deal with Hilbert space \(L_2(V)\). This space we choose as the domain of operator \(\mathcal{R}\).

One can verify that the following important identity holds:

\[(\mathcal{R} \mathcal{X}_1, \mathcal{X}_2) = (\mathcal{X}_1, \mathcal{R} \mathcal{X}_2).\]  

(22)

It means that operator \(\mathcal{R}\) is self-adjoint in \(L_2(V)\), and eigenvalue equation holds

\[\mathcal{R} \mathcal{X}_n (\mathbf{r}) = \kappa_n \mathcal{X}_n (\mathbf{r}), \quad \mathcal{X}_n (\mathbf{r}) = \text{col} (\mathbf{E}_n (\mathbf{r}), \mathbf{H}_n (\mathbf{r})).\]  

(23)

Operator equation (23) was well studied in the book [7]. Here is some results. Spectrum of the eigenvalues \(\{\kappa_n\}\) is discrete because \(V\) is finite. All the eigenvalues \(\kappa_n\) are located on real axis (because \(\mathcal{R}\) is self-adjoint) symmetrically with respect to point \(\kappa = 0\) : when \(\kappa_{+n} > 0\) is an eigenvalue of \(\mathcal{R}\), then \(\kappa_{-n} = -\kappa_{+n}\) is its eigenvalue as well. All the eigenvalues \(\kappa_{\pm n} \neq 0\) may have only a finite power of degeneracy. Number \(\kappa = 0\) is eigenvalue of \(\mathcal{R}\) as well, but it has infinite power of degeneracy. Eigenvectors of operator \(\mathcal{R}\) have the following forms:

\[
\mathcal{X}_{\pm n} = \begin{pmatrix} \mathbf{E}_n \\ \pm \mathbf{H}_n \end{pmatrix}, \quad \mathcal{X}_\alpha = \begin{pmatrix} \nabla \Phi_\alpha \\ 0 \end{pmatrix}, \quad \mathcal{X}_\beta = \begin{pmatrix} 0 \\ \nabla \Psi_\beta \end{pmatrix},
\]  

(24)

where eigenvector \(\mathcal{X}_{+n}\) corresponds to the eigenvalue \(\kappa_{+n} \neq 0\) but \(\mathcal{X}_{-n}\) corresponds to symmetrical eigenvalue \(\kappa_{-n} = -\kappa_{+n}\). Two infinite sets of eigenvectors \(\{\mathcal{X}_\alpha\}\) and \(\{\mathcal{X}_\beta\}\) correspond to single eigenvalue \(\kappa_0 = 0\). Their potentials \(\Phi_\alpha\) and \(\Psi_\beta\) are specified via just the same boundary eigenvalue problems for Laplacian as in Eqs. (11) and (12). All the eigenvectors \(\mathcal{X}_p\) and \(\mathcal{X}_q\), corresponding to distinct eigenvalues \(\kappa_p\) and \(\kappa_q \neq \kappa_p\), are orthogonal in the sense of inner product (21) as \((\mathcal{X}_p, \mathcal{X}_q) = 0\). Orthogonality \((\mathcal{X}_\alpha^\dagger, \mathcal{X}_\beta^\dagger) = 0\) is evident.

Operator equation (23) written out with respect to components \(\mathbf{E}_n, \mathbf{H}_n\) of the eigenvectors \(\mathcal{X}_{\pm n}\) from (24) looks as follows (with notation \(\kappa_{+n} \equiv \kappa_n > 0\))

\[
\begin{align*}
\begin{cases}
\text{rot} \mathbf{H}_n = i \kappa_n \varepsilon_0 \varepsilon \mathbf{E}_n, & (\mathbf{n} \cdot \mathbf{H}_n) |_{S} = 0, \\
\text{rot} \mathbf{E}_n = -i \kappa_n \mu_0 \mu \mathbf{H}_n, & [\mathbf{n} \times \mathbf{E}_n] |_{S} = 0.
\end{cases}
\end{align*}
\]  

(25)
Of course, boundary condition \((\mathbf{n} \cdot \mathbf{H}_n)|_S = 0\) follows from \([\mathbf{n} \times \mathbf{E}_n]|_S = 0\) and vice versa while \(\kappa_n \neq 0\). However, the differential equations from problem (25) and \([\mathbf{n} \times \mathbf{E}_n]|_S = 0\) give jointly eigenvalue problem (9) with \(\kappa_n \equiv \omega_n'\). The same equations (25) and \((\mathbf{n} \cdot \mathbf{H}_n)|_S = 0\) give jointly (10) with \(\kappa_n \equiv \omega_n''\).

Thus, completeness of time harmonic field decompositions (8) with frequency dependent modal amplitudes guarantees completeness of decompositions (14) for field \(\mathcal{E}(\mathbf{r}, t), \mathcal{H}(\mathbf{r}, t)\) with time dependent modal amplitudes. We have proved just this fact and kept \(\partial_t\) in Maxwell’s equations: in their operator form (19).

In closing, we should supplement the eigenvalue problems (9)–(12) with proper normalizing conditions introducing them as follows

\[
(\varepsilon_0 \varepsilon / V) \int_V |E_{n}^p|^2 \, dv = 1, \quad (\mu_0 \mu / V) \int_V |H_{n}^p|^2 \, dv = 1, \\
(\kappa_{\alpha}^2 \varepsilon_0 \varepsilon / V) \int_V |\Phi_{\alpha}|^2 \, dv = 1, \quad (\nu_{\beta}^2 \mu_0 \mu / V) \int_V |\Psi_{\beta}|^2 \, dv = 1, \tag{26}
\]

where superscript \(p\) means proper prime or double prime.

Thus, all the vectors are specified completely in Eqs. (14). They constitute jointly a modal basis*. The only problem remains now: to find out the scalar modal amplitudes depending on time in those decompositions (14).

### 3. CAVITY EVOLUTIONARY EQUATIONS

Hilbert space \(L_2\), introduced above via inner product (21), is the space of solutions for our problem. Problem for the modal amplitudes can be obtained via projecting Maxwell’s equations onto the same modal basis in \(L_2\) with its elements (24). This procedure can be implemented with using identity (22).

Let us put \(\mathcal{X}_1 = \mathcal{X}(\mathbf{r}, t)\) in (22). For the vector \(\mathcal{R} \mathcal{X}_1\) in this identity (i.e., for \(\mathcal{R} \mathcal{X}_1 \equiv \mathcal{R} \mathcal{X}(\mathbf{r}, t)\)) we can use Maxwell’s equations (in their form (19). where \(\partial_t\) is conserved!) as a direct formula: \(\mathcal{R} \mathcal{X}(\mathbf{r}, t) = i (\partial_t \mathcal{X}(\mathbf{r}, t) + \mathcal{F}(\mathbf{r}, t))\). As the vector \(\mathcal{X}_2\) in (22), we can take any eigenvector \(\mathcal{X}_n : \mathcal{X}_2 = \mathcal{X}_n\), and use eigenvalue equation (23)

---

* In book[7], we used Weyl’s theorem[14] about orthogonal decompositions of Hilbert space \(L_2(V)\) in solutions of boundary eigenvalue problems for Laplacian. In terms of notations (24), this theorem defines a basis as \(L_2(V) = \{\mathcal{X}_{\pm n}\} \oplus \{\mathcal{X}_{\alpha}\} \oplus \{\mathcal{X}_{\beta}\}\), where \(\oplus\) means direct summation of the mutually orthogonal subspaces. This identity gives presentation for \(\mathcal{X}(\mathbf{r}, t)\) as formal series \(\mathcal{X}(\mathbf{r}, t) = \sum_{n} c_n(t) \mathcal{X}_n(\mathbf{r})\). It can be rearranged to the form of Eqs. (14).
as a direct formula again: \( \Re \mathbf{x}_n = \kappa_n \mathbf{x}_n \). It supplies with

\[
\frac{d}{dt}(\mathbf{x}(r,t), \mathbf{x}_n) + (\mathcal{F}(r,t), \mathbf{x}_n) = -i \kappa_n (\mathbf{x}(r,t), \mathbf{x}_n), \quad n = 0, 1, 2, \ldots
\]

(27)

In Eq. (27), we should use definition (17) for vector \( \mathbf{x}(r,t) \) with modal decompositions (14) for its components. To use up all possible eigenvectors \( \mathbf{x}_n \)'s in Eq. (27), it is enough to take only four ones, which are composed of the eigensolutions of (9) – (12) subjected to the normalizing conditions (26), namely:

\[
\mathbf{x}'_n = \begin{pmatrix} \mathbf{E}'_n \\ \mathbf{H}'_n \end{pmatrix}, \quad \mathbf{x}''_n = \begin{pmatrix} \mathbf{E}''_n \\ \mathbf{H}''_n \end{pmatrix}, \quad \mathbf{x}^\upalpha = \begin{pmatrix} \nabla \Phi^\alpha \\ 0 \end{pmatrix}, \quad \mathbf{x}^\downbeta = \begin{pmatrix} 0 \\ \nabla \Psi_\beta \end{pmatrix},
\]

where subscripts \( n, \alpha, \beta = 1, 2, 3, \ldots \) play role of free spectral parameters. Eigenvectors \( \mathbf{x}'_n \) and \( \mathbf{x}''_n \) correspond to \( \kappa_n = \omega'_n \) and \( \kappa_n = \omega''_n \), respectively: see Eqs. (9), (10). Eigenvectors \( \mathbf{x}^\upalpha \) and \( \mathbf{x}^\downbeta \) correspond to single eigenvalue \( \kappa_0 = 0 \).

Simple but tiresome manipulations with inner products from Eq. (27), and with using the orthonormal conditions give in the end four sets of uncoupled evolutionary equations for the time dependent modal amplitudes as

\[
\frac{d}{dt} \begin{pmatrix} e'_n \\ h'_n \end{pmatrix} + \begin{pmatrix} 2\gamma & i\omega'_n \\ i\omega'_n & 0 \end{pmatrix} \begin{pmatrix} e'_n \\ h'_n \end{pmatrix} = -\begin{pmatrix} j'_{en}(t) \\ j'_{mn}(t) \end{pmatrix},
\]

(28)

\[
\frac{d}{dt} \begin{pmatrix} e''_n \\ h''_n \end{pmatrix} + \begin{pmatrix} 2\gamma & i\omega''_n \\ i\omega''_n & 0 \end{pmatrix} \begin{pmatrix} e''_n \\ h''_n \end{pmatrix} = -\begin{pmatrix} j''_{en}(t) \\ j''_{mn}(t) \end{pmatrix},
\]

(29)

\[
\frac{d}{dt} a_\alpha + 2\gamma a_\alpha = -i_{e\alpha}(t), \quad \frac{d}{dt} b_\beta = -i_{m\beta}(t),
\]

(30)

where \( \gamma = \sigma/2\varepsilon_0 \varepsilon \). The force terms in Eqs. (28) – (30) are projections onto the same basis elements of the function \( \mathcal{F} \) mentioned in Eqs. (27), (19) and (1a):

\[
\begin{align*}
\ j_{en}^p(t) &= \frac{1}{V} \int_V \mathcal{J}_e(r,t) \cdot \mathbf{E}_n^{p*} dv, \quad j_{mn}^p(t) = \frac{1}{V} \int_V \mathcal{J}_m(r,t) \cdot \mathbf{H}_n^{p*} dv; \\
\ i_{e\alpha}(t) &= \frac{1}{V} \int_V \mathcal{J}_e(r,t) \cdot \nabla \Phi_\alpha^{*} dv, \quad i_{m\beta}(t) = \frac{1}{V} \int_V \mathcal{J}_m(r,t) \cdot \nabla \Psi_\beta^{*} dv,
\end{align*}
\]

(31)

where superscript \( p \) stands for proper prime or double prime.

In the initial conditions (4), let's put \( \mathcal{E}(r,0) = 0, \mathcal{H}(r,0) = 0 \). This is equivalent to choice of initial conditions for the modal
amplitudes as
\[
\begin{align*}
    e'_n(0) &= 0, & e''_n(0) &= 0, & a_\alpha(0) &= 0, \\
    h'_n(0) &= 0, & h''_n(0) &= 0, & b_\beta(0) &= 0.
\end{align*}
\] (32)

Eqs. (30) for amplitudes of the irrotational modes have evident solutions
\[
a_\alpha(t) = -e^{-2\gamma t} \int_0^t e^{2\gamma t'} i_{\epsilon \alpha}(t') \, dt',
\quad b_\beta(t) = -\int_0^t i_{m \beta}(t') \, dt'.
\] (33)

Hence, irrotational part of field (14) is not at all a static as it gives MCA. Factually, it varies in time in a proper way under driving the signal applied.

Time dependence of the solenoidal modes is defined by solutions of Eqs. (28) and (29). Mathematically, they are identical. Therefore, we can release them from unnecessary subscripts and superscripts and present them both as follows
\[
\frac{d}{dx} \begin{pmatrix} e(x) \\ h(x) \end{pmatrix} + \begin{pmatrix} 2g & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e(x) \\ h(x) \end{pmatrix} = -\begin{pmatrix} j_e(x) \\ j_m(x) \end{pmatrix},
\] (34)

where \( i \) is the imaginary unit, as before, and
\[
x = \omega_n^p t, \quad g = \gamma / \omega_n^p, \\
\quad j_e(x) = j_{e n}^p(t) / \omega_n^p, \quad j_m(x) = j_{m n}^p(t) / \omega_n^p.
\] (35)

Variable \( x \) is now dimensionless time and \( g \) is dimensionless lossy parameter.

4. EXPLICIT SOLUTIONS AND NUMERICAL EXAMPLES

Let’s introduce new notation as follows
\[
Y(x) = \begin{pmatrix} e(x) \\ h(x) \end{pmatrix}, \quad Q = \begin{pmatrix} 2g & i \\ i & 0 \end{pmatrix}, \quad f(x) = \begin{pmatrix} j_e(x) \\ j_m(x) \end{pmatrix}.
\] (36)

Then equation (34) with initial conditions (32) jointly gives Cauchy problem as
\[
\frac{d}{dx} Y(x) + Q Y(x) = -f(x), \quad Y(0) = 0.
\] (37)
The force term at differential equation (37) is specified by the functions of impressed sources (1a). So, \( f(x) \neq 0 \) within \( 0 \leq x \leq X \), where \( X = T \omega_n^p \), otherwise \( f(x) \equiv 0 \). Hence, solution to the problem (37) is sum of two addends:

\[
Y(x) = [H(x) - H(x - X)] Y_1(x) + H(y) Y_2(y),
\]

where \( y \equiv x - X \), \( H(\cdot) \) is Heaviside step function, and

\[
\frac{d}{dx} Y_1(x) + Q Y_1(x) = -f(x), \quad Y_1(0) = 0, \quad \text{while } 0 \leq x \leq X; \quad (39)
\]

\[
\frac{d}{dy} Y_2(y) + Q Y_2(y) = 0, \quad Y_2(0) = Y_1(X), \quad \text{while } y \geq 0. \quad (40)
\]

Solution (38) satisfies the principle of causality (6) automatically.

### 4.1. Analytical Solutions in the General Case

Formal solutions to the problems (39) and (40) can be written easily as

\[
Y_1(x) = e^{-Qx} \int_0^x e^{Qx'} f(x') \, dx', \quad Y_2(y) = e^{-Qy} Y_1(X),
\]

where \( \exp(-Qx) \) is a matrix of the same order as \( Q \). In turn, \( \exp(Qx') \) is the matrix inverse to \( \exp(-Qx) \), which can be calculated via formula

\[
e^{Qx'} = e^{-Qx}|_{x=-x'}.
\]

So, we have the only problem now: how to calculate efficiently that exponential with matrix argument? About 50 years ago, Hayashi proposed [15] a convenient procedure for calculation explicitly of any function with argument involving a constant matrix. As regards our exponential, it yields

\[
e^{-Qx} = \frac{e^{-gx}}{\varpi} \left( \begin{array}{cc}
\cos(\varpi x + \delta) & -i \sin(\varpi x) \\
-i \sin(\varpi x) & \cos(\varpi x - \delta)
\end{array} \right),
\]

where \( \varpi = \sqrt{1 - g^2} \), \( \delta = \arcsin(g) = \arccos(\varpi) \); \( g \) see in Eqs. (35). Term \( \varpi x \) at the argument involves time \( t \) as \( \varpi x = \omega_n^p \varpi t \). Hence, factor \( \omega_n^p \varpi \) specifies a frequency; let’s denote it as \( \omega \). Coefficient \( \omega_n^p \) with \( (p) \equiv (') \) or \( (\prime) \) specifies eigenfrequencies of a cavity in the eigenvalue problems (9) and (10). Factor \( \varpi \) is specified by ohmic lossy parameter \( \sigma \). So,

\[
\omega = \omega_n^p \varpi \equiv \omega_n^p \sqrt{1 - (\sigma/2\varepsilon_0\varepsilon \omega_n^p)^2}
\]
is the frequency of oscillations in a cavity loaded with a lossy medium. Substitution of \( f (x') \) (see (36)) at the integrand in (41) gives two "quadrature source functions" as

\[
J_e (x) = e^{-gx} \int_0^x e^{gx'} \left[ j_e (x') \cos (wx' - \delta) + i j_m (x') \sin (wx') \right] dx',
\]

\[
J_m (x) = e^{-gx} \int_0^x e^{gx'} \left[ j_m (x') \cos (wx' + \delta) + i j_e (x') \sin (wx') \right] dx'.
\]

(45)

Note in passing that it is convenient to define electric current \( J_e (r, t) \) and magnetic current \( J_m (r, t) \) in (1a) as real functions. Then quadrature formulas (45) give real function \( J_e (x) \) and pure imaginary function \( J_m (x) \).†

First formula in Eqs. (41) gives explicit solutions for the modal amplitudes††

\[
e_1 (x) = \frac{[J_e (x) \cos (wx + \delta) - i J_m (x) \sin (wx)]}{\omega^2},
\]

\[
h_1 (x) = \frac{[J_m (x) \cos (wx - \delta) - i J_e (x) \sin (wx)]}{\omega^2},
\]

(46)

where dimensionless time \( x \) varies within interval \( 0 \leq x \leq X \). These amplitudes define forced oscillations in the cavity under consideration. At the moment of time \( t = T \) (i.e., \( x = X \)), external signals cease their action. But oscillations in the cavity continue with modal amplitudes defined by the second formula in Eqs. (41) that yields evident solution as

\[
e_2 (y) = e^{-gy}[e_1 (X) \cos (wy + \delta) - i h_1 (X) \sin (wy)]/\omega, \quad y \geq 0,
\]

\[
h_2 (y) = e^{-gy} [h_1 (X) \cos (wy - \delta) - i e_1 (X) \sin (wy)]/\omega, \quad y \geq 0.
\]

(47)

4.2. Simple Explicit Solutions for a Particular Case

Functions \( J_e (x) \) and \( J_m (x) \) can be calculated explicitly for various particular cases of the signals. Let's consider one example when typical sinusoidal signal is applied with a constant amplitude \( A \), but with finite duration within \( 0 \leq t \leq T \). In view of notation (35), it should be introduced as

\[
j_e (t) = [H (t) - H (t - T)] A \sin (\Omega t)/\omega_n, \quad j_m (t) \equiv 0.
\]

(48)

Calculations of formulas (45)–(47) for the case (48) give a result expressed in elementary functions. To facilitate physical reading the

† See definitions (31) for \( j_e (x') \) and \( j_m (x') \), where \( E_n^P \) is a real function but \( H_n^P \) is pure imaginary one in accordance with their definitions in Eqs. (9) and (10).

†† Amplitude \( e_1 (x) \) is real function but \( h_1 (x) \) is pure imaginary one.
final results, we present them with using original notation and formula (44).

While time of observation \( t < 0 \), all the modal amplitudes are identical zero. While \( 0 \leq t \leq T \), they look like result of a competition\(^\#\) between forced oscillation with frequency \( \Omega \) and loaded oscillation with frequency \( \omega \), namely:

\[
e(t)_{t<0} = A \frac{-2 \sqrt{1 - (\gamma/\omega_n)^2}}{2\sqrt{\gamma^2 + (\Omega - \omega)^2}} \left[ \sin(\Omega t + \varphi) - e^{-\gamma t} \sin(\omega t + \varphi) \right] + O(1),
\]

\[
h(t)_{t<0} = i A \frac{-2 \sqrt{1 - (\gamma/\omega_n)^2}}{2\sqrt{\gamma^2 + (\Omega - \omega)^2}} \left[ \cos(\Omega t - \phi) - e^{-\gamma t} \cos(\omega t - \phi) \right] + O(1).
\]

Phase shift \( \varphi = \delta - \phi \) is specified by two addends of different physical nature as

\[
\delta = \arcsin(\gamma/\omega_n^2), \quad \gamma = \sigma/2\varepsilon_0\varepsilon; \quad \phi = \arcsin\left[ (\Omega - \omega) / \sqrt{\gamma^2 + (\Omega - \omega)^2} \right].
\]

When the signal applied is turned off at \( t = T \), the modes continue their oscillations after freely with time dependent amplitudes as

\[
e(t)_{t>T} = A \frac{-2 \sqrt{1 - (\gamma/\omega_n)^2}}{2\sqrt{\gamma^2 + (\Omega - \omega)^2}} \left( e^{-\gamma(t-T)} - e^{-\gamma t} \right) \sin(\omega t + \tilde{\varphi}) + O(1),
\]

\[
h(t)_{t>T} = i A \frac{-2 \sqrt{1 - (\gamma/\omega_n)^2}}{2\sqrt{\gamma^2 + (\Omega - \omega)^2}} \left( e^{-\gamma(t-T)} - e^{-\gamma t} \right) \cos(\omega t + \tilde{\varphi}) + O(1),
\]

where \( \tilde{\varphi} = \varphi + (\Omega - \omega)T \).

In the case of resonance, when \( \Omega = \omega \), these formulas become simpler as

\[
e(t)_{t<T} = A \frac{1}{2} \frac{1 - e^{-\gamma t}}{\gamma \sqrt{1 - (\gamma/\omega_n)^2}} \sin(\Omega t + \delta) + O(1),
\]

\[
h(t)_{t<T} = i A \frac{1}{2} \frac{1 - e^{-\gamma t}}{\gamma \sqrt{1 - (\gamma/\omega_n)^2}} \cos(\Omega t) + O(1).
\]

\[
e(t)_{t>T} = A \frac{1}{2} \frac{e^{-\gamma(t-T)} - e^{-\gamma t}}{\gamma \sqrt{1 - (\gamma/\omega_n)^2}} \sin(\Omega t + \delta) + O(1),
\]

\[
h(t)_{t>T} = i A \frac{1}{2} \frac{e^{-\gamma(t-T)} - e^{-\gamma t}}{\gamma \sqrt{1 - (\gamma/\omega_n)^2}} \cos(\Omega t + O(1).
\]

\(^\#\) or like a "dynamic interference"
While lossy parameter $\gamma$ tends to zero with $\sigma \to 0$, but $t$ and $T$ are finite, then

$$\lim_{\sigma \to 0} \frac{1 - e^{-\gamma t}}{\gamma \sqrt{1 - (\gamma/\omega_n)^2}} = t, \quad \lim_{\sigma \to 0} \frac{e^{-\gamma(T-t)} - e^{-\gamma t}}{\gamma \sqrt{1 - (\gamma/\omega_n)^2}} = T, \quad \lim_{\sigma \to 0} \delta = 0.$$ 

Hence, $e(t), h(t) \sim t$ within $0 \leq t \leq T$. The amplitudes oscillate freely after (i.e., while $t > T$) with frequency $\omega = \omega_n^p$ and stationary their amplitudes proportional to $T$; see also numerical results further in Figs. 1 and 2.

**Figure 1.** Resonance of $e(x) : k = 1, g = 0, N = 20$.

**Figure 2.** Resonance of $h(x) : k = 1, g = 0, N = 20$. 
4.3. Numerical Examples

Let's present duration of the sinusoidal signal applied as $X = 2\pi N$ and choose for calculations value $N = 20$ in order to obtain oscillations of the modal amplitudes distinguishable on pictures. The signal applied with amplitude $A = 1$ will be exhibited as well on all the pictures below with darken line. Lighten line will be used for demonstration of the the modal amplitudes. For calculations of the modal amplitudes $e(x)$ and $h(x)$ as functions of dimensionless time $x$, we used formulas (38), (45)–(48) and parameter $k = \Omega/\omega$; $\omega$ see in Eq. (44).

Figs. 1 and 2 show resonance ($k = 1$) in a cavity filled with loss-free medium ($g = 0$). Modal amplitude of electric field $e(x)$ and magnetic field $h(x)$ start their oscillations with $e(0) = 0$, $h(0) = 0$ together. However, $h(x)$ obtains additional phase shift $\pi/2$ very soon, within $0 \leq x \leq 3\pi$, and keep it after. Inasmuch as $h(x)$ dependencies are very similar to $e(x)$ ones for the same set of the computation parameters, all the pictures for $h(x)$ we omit further.

Presence of ohmic losses in the medium leads to effect of satiation in amplitude of resonant vibrations: see in Fig. 3.

![Graph of oscillation](image)

**Figure 3.** Resonance of $e(x)$ in lossy medium: $k = 1$, $g = 0.05$, $N = 20$.

When the resonance condition is disturbed: $k \neq 1$, then modal oscillations suffer beating within $0 \leq x \leq X$. Vibrations continue after ($x > X$) as free oscillations. It is possible to find a specific resonance condition for beating in hollow loss-free cavity. When $e(X) = 0$ and $h(X) = 0$ (while $k \neq 1$, $g = 0$), then free oscillations are canceled at $x \geq X$ in accordance with Eqs. (47). In closing, we demonstrate beating in a cavity filled with a lossy medium when duration $N$ of the signal applied is much more then in previous cases.
Figure 4. Beating in $e(x)$ for $k = 1.075; \ g = 0, \ N = 20$.

Figure 5. "Resonant" beating in $e(x)$ for $k = 1.05; \ g = 0, \ N = 20$.

Figure 6. Beating in $e(x)$ for $k = 1.05; \ g = 0.005, \ N = 200$. 
5. CONCLUDING REMARKS

A cavity is considered as an item placed in some operating microwave system. A given time-domain signal supplied for processing in the cavity turns it into a time-variant physical system. Every real time-domain signal has a beginning and end. It requires supplementing the formulation of the problem with the principle of causality and solving it with using appropriate time-domain method.

The problem has been solved with using the *Evolutionary Approach to Electromagnetics* proposed recently. One can find here implementation of the EAE that is available for studying a wide class of such time-variant cavity systems. Powers of freedom for choice of a concrete system within this class are (i) geometrical form and dimensions of a cavity; (ii) electromagnetic parameters of a medium which is placed in a chosen cavity; (iii) time-domain pattern of a signal supplied in the cavity filled with a chosen medium.

1. Presentation of cavity fields has been derived and justified as the classical eigenmodal decompositions but with time dependent modal amplitudes and with keeping time derivative in Maxwell’s equations.

2. Projecting Maxwell's equations (with $\partial_t$) onto the cavity modal basis gives *evolutionary* (i.e., with time derivative) ordinary differential equations for the modal amplitudes supplemented with initial conditions (*Cauchy problem*).

3. Solutions to the Cauchy problem satisfy the causality principle. Solution of the Cauchy problem has been obtained in closed form (in quadratures) for a signal with arbitrary time-domain pattern different from zero within $0 \leq t \leq T$ ($T > 0$ is arbitrary), otherwise identical zero.

Habitual practical case is considered in detail when signal supplied is like $[H(t) - H(t - T)] \sin(\Omega t)$, where $H(\cdot)$ is Heaviside step function. In this case, the modal amplitudes are obtained in elementary functions. Resonance is considered when $\Omega = \omega$, where $\omega$ is frequency of oscillations of a mode in a cavity loaded with a lossy medium. In resonance, its modal amplitude within $0 \leq t \leq T$ is product of fast sinusoidal oscillations with the frequency $\Omega$ and factor $\left(1 - e^{-\gamma t}\right)/\gamma$, where $\gamma = \sigma/2\varepsilon_0\varepsilon$ is the lossy parameter of a medium that fills the cavity. When conductivity of the medium $\sigma \to 0$ but $T$ is finite, this modal amplitude grows up in resonance proportionally to time of observation $t$. 
REFERENCES


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