Surplus of energy for time-domain waveguide modes

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Abstract

In this study, a problem for electromagnetic fields produced by a time dependent source function in a waveguide with perfect electric conductor surfaces is considered by a direct analytical time-domain method called \textit{Evolutionary Approach to Electromagnetics}. The previous works are revisited for energy and surplus of energy of time-domain modes in the waveguides. A complete set of transverse electric and transverse magnetic waveguide modes is obtained in time-domain, directly. Every field component of the modes is product of two factors: First one is a vector function of transverse waveguide coordinates which corresponds to a modal basis problem. It is specified via well studied Dirichlet and Neumann boundary eigenvalue problems. Physically, these vector functions are distributions of the modal force lines in the waveguide cross-section. The second one is a scalar function corresponds to a time-dependent modal amplitude problem. This is obtained as the solution of Klein-Gordon equation depend on the waveguide’s longitudinal coordinate and time. Consequently, the problem of time-domain signal propagation in the waveguide is solved analytically in compliance with a causality principle. The graphical results are shown for the cases when the energy and surplus of the energy for the waveguide time-domain waveguide modes are represented via the first kind Bessel functions of semi-integer order.

Keywords: Time domain waveguide mode; Klein-Gordon equation; Energy; Surplus of the energy

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1. Introduction

Time-harmonic waveguide modes are usually interpreted for signal transmission along waveguides. However, this model has two essential physical drawbacks. Firstly, the time-harmonic signals are non-casual. It means that their propagation starts at time $t = -\infty$ and continues up to time $t = \infty$. Secondly, these signals have frequency bandwidth equal to zero. Therefore, this is not a satisfactory model for the signal transmission problems in the waveguides. Moreover, generation of the transient response can be proposed by a Fourier transform; however, it is a troublesome problem because the frequency response at many frequencies must be known with a high degree of precision. The initial conditions cannot be satisfied, simply. These can cause inaccurate calculations especially for the late time part of the transients. Thus this technique will be questionable.
Leaving aside the works based on the synthesizing realistic signals via continual superposition of the time-harmonic waves by using a Fourier or Laplace transform, it seems that one of the first noticeable approaches for direct time domain solutions of electromagnetic problems was developed within the framework of four-dimensional relativistic formalism in electrodynamics [1]. Then another alternative approach called the Evolutionary Approach to Electromagnetics (EAE) suggested in 80s was proposed for the direct time-domain theory of the cavity and waveguide modes. Important published works regarded to the EAE method for the time-domain waveguide problems were given in the literature [2]-[7]. The other set of important publications on this topic is based on the different techniques [8]-[12].

Knowledge about the properties of hyperbolic kind Partial Differential Equation (PDE) suggests alternative attacks to new classes of the waveguide problems. The proposed approach leads to PDEs for the modal amplitudes in the time-domain. In this article, the previous works [2]-[7] obtained within the framework of the EAE method are revisited, however in the sense of energetic considerations for further analysis. The EAE method is based on the solution of sequential two autonomous problems. The first one is a “modal basis problem” corresponds to well-studied Dirichlet and Neumann boundary eigenvalue problems. This involves two complete sets as the Transverse-Electric (TE) and Transverse-Magnetic (TM) modes. Because the generating scalar potentials for the TE and TM modes are actually the same as the time-harmonic modes, one can freely use the methods, which have been developed in the frequency domain, and use as well even ready results obtained for the complex waveguide configurations. The second one is a “time-dependent modal amplitude problem” corresponds to Klein-Gordon Equation (KGE) with the axial coordinate and time. Our main effort is addressed to obtain the analytical solution to the KGE leading to the time-dependent modal amplitudes unlike to those in the time-harmonic waves.

This article is composed as follows: In Section II, the fundamental ideas of the EAE method for the time-domain waveguide problems are given. The properties of symmetry in the KGE disclosed within the framework of group theory are discussed. It is explained how to use these symmetry properties in order to obtain the direct analytical time-domain solution for the time-dependent waves in the waveguide. In Section III, the general properties of the waveguide time-domain modes such as completeness and invariability under Lorentz transformation are considered. The initial conditions for the modal amplitudes are exhibited. The causality principle is imposed on the time-domain modes. In Section IV, conservation of energy law is given in differential form for the time-domain modes, Energy and surplus of the energy are considered both via the first kind Bessel functions of semi-integer order. The graphical examples are shown. In the last section, discussion and conclusion part are given.

2. Problem of time-domain modes

A hollow (i.e., medium-free) waveguide with its cross-section domain S bounded by a closed singly connected contour L is considered. It is supposed that L has enough smooth shape which implies that none of possible inner angles of L (i.e., being measured within S) exceeds π and the cross section S maintains its form and size along the waveguide axis OZ. Our aim is to solve the modal fields for the TE and TM modes which are a particular solution to the system of Maxwell’s equations with the time derivative given as

\[
\nabla \times \mathbf{E}(\mathbf{R}, t) = -\mu_0 \frac{\partial}{\partial t} \mathbf{H}(\mathbf{R}, t) , \quad \nabla \times \mathbf{H}(\mathbf{R}, t) = \varepsilon_0 \frac{\partial}{\partial t} \mathbf{E}(\mathbf{R}, t)
\]

where \(\mathbf{E}(\mathbf{R}, t)\) and \(\mathbf{H}(\mathbf{R}, t)\) are the electric and magnetic fields, respectively. \(\varepsilon_0\) and \(\mu_0\) are dielectric and magnetic constants for free-space, respectively. Because the fields will be
excited by an initial condition technique, the source term is not considered in the Maxwell’s equations. The vector \( \mathbf{R} \) within the waveguide volume denotes an observation point. \( t \) is observation time. Let’s introduce a right-handed triplet of the mutually orthogonal unit vectors \((\mathbf{z}, \mathbf{l}, \mathbf{n})\) where \( \mathbf{z} \times \mathbf{l} = \mathbf{n} \). The unit vector \( \mathbf{z} \) and \( \mathbf{l} \) are tangential to the axis \( O_z \) and contour \( L \), respectively. The unit vector \( \mathbf{n} \) is outward normal to the cross-section of domain \( S \).

Let’s decompose the vector \( \mathbf{R} \) and Nabla operator \( \nabla \) onto their transverse and longitudinal parts as
\[
\mathbf{R} = \mathbf{r} + z\mathbf{z}, \quad \nabla = \nabla_\perp + z\partial_z
\]
where the projection \( \mathbf{r} \) is a position vector within the domain \( S \) and \( \nabla_\perp \) is the transverse Laplacian operator.

Subject of our study is real-valued electromagnetic fields specified by the electric and magnetic field strength vectors \( \mathbf{E}_m(\mathbf{R}, t) \) and \( \mathbf{H}_m(\mathbf{R}, t) \), respectively. Separate these vectors onto their transverse and longitudinal parts similarly to performed in Eq. (2), i.e.,
\[
\mathbf{E}_m(\mathbf{R}, t) = \mathbf{E}(\mathbf{r}, z, t) + z\mathbf{E}_z(\mathbf{r}, z, t) \\
\mathbf{H}_m(\mathbf{R}, t) = \mathbf{H}(\mathbf{r}, z, t) + z\mathbf{H}_z(\mathbf{r}, z, t)
\]
where \( m = 1, 2, \ldots \). Because the waveguide surface is supposed to have physical properties of the perfect electric conductor, the following boundary conditions hold over the waveguide surface
\[
n \cdot \mathbf{H}_m(\mathbf{R}, t)|_L = 0, \quad l \cdot \mathbf{E}_m(\mathbf{R}, t)|_L = 0, \quad z \cdot \mathbf{E}_m(\mathbf{R}, t)|_L = 0.
\]

2.1. The modal basis problem
2.1.1. TE time-domain modes

Let’s consider the Neumann boundary eigenvalue problem for the operator \( \nabla_\perp^2 \) as
\[
(\nabla_\perp^2 + \upsilon_m^2)\psi_m(\mathbf{r}) = 0, \quad \frac{\partial \psi_m(\mathbf{r})}{\partial n}|_L = 0, \quad \frac{\upsilon_m^2}{S} \int_S |\psi_m(\mathbf{r})|^2 \, ds = 1 \ N
\]
where \( \partial_n = n \cdot \nabla_\perp \) is the normal derivative on the contour \( L \). \( \upsilon_m^2 > 0 \) and \( m = 1, 2, 3, \ldots \) are the eigenvalues and their regulation numbers of position on a real axis in the increasing order of their numerical values. The potentials \( \psi_m(\mathbf{r}) \) are the eigenvectors of the corresponding eigenvalues. Force dimension \( N \) (i.e., newton) in Eq. (5) is involved in order to provide the required physical dimensions for the field vector components of \( \mathbf{E}_m \) and \( \mathbf{H}_m \) as \( \upsilon m^{-1} \) and \( Am^{-1} \), respectively.

For the eigenvalue \( \upsilon_0^2 = 0 \), the problem (5) will have the following form
\[
\nabla_\perp^2 \psi_0(\mathbf{r}) = 0, \quad \frac{\partial \psi_0(\mathbf{r})}{\partial n}|_L = 0
\]
where the function \( \psi_0(\mathbf{r}) \) is a harmonic function and its value is distinct from zero. The minimum-maximum theorem for the harmonic functions yields that \( \psi_0(\mathbf{r}) = C \) where \( \mathbf{r} \in L + S \) and \( C \) is an arbitrary constant.

Every particular solution \( \psi_m(\mathbf{r}) \) to the Neumann problem (5) generates the TE time-domain modal fields with the components as

\[
E_{e,m}^h = 0
\]

\[
\nu_m^{-1} E_m^h = \left( -\partial_{\nu_m} h_m(z,t) \right) \left[ -\frac{1}{\varepsilon_0} \mu_0 \omega A_m^T \nabla_{\perp} \psi_m(\mathbf{r}) \times \mathbf{z} \right]
\]

\[
\nu_m^{-1} H_m^h = \left( \partial_{\nu_m} h_m(z,t) \right) \left[ \frac{1}{\mu_0} \epsilon_0 \omega A_m^T \nabla_{\perp} \psi_m(\mathbf{r}) \right]
\]

\[
\nu_m^{-1} H_m^z = \left( h_m(z,t) \right) \left[ \mu_0^{-1} \epsilon_0 \omega A_m^T \psi_m(\mathbf{r}) \right]
\]

where \( \partial_{\nu_m} = (1/c \nu_m) \partial / \partial t \), \( \partial_{\nu_m} = (1/\nu_m) \partial / \partial z \) and \( c = 1/\sqrt{\varepsilon_0 \mu_0} \). Specially, the potential \( \psi_0(\mathbf{r}) \) generates a one-component modal field as

\[
E_0(\mathbf{r}, z, t) = 0 \quad , \quad H_0(\mathbf{r}, z, t) = zC
\]

where dimension \( Am^{-1} \) should be assigned to constant \( C \).

The potential \( h_m(z,t) \) in Eq. (7) is governed by Klein-Gordon Equation (KGE)

\[
\left( \partial_{\nu_m}^2 - \partial_{\nu_m}^2 + 1 \right) h_m(z,t) = 0
\]

which is known as a generalized wave equation [5], [6].

### 2.1.2. TM time-domain modes

As similar to the problem of the TE time-domain modes, the Dirichlet boundary eigenvalue problem for the operator \( \nabla_{\perp}^2 \) can be stated as follows

\[
\left( \nabla_{\perp}^2 + \kappa_m^2 \right) \phi_m(\mathbf{r}) = 0 \quad , \quad \phi_m(\mathbf{r}) \bigg|_{L} = 0 \quad , \quad \frac{\kappa_m^2}{S} \int_{S} |\phi_m(\mathbf{r})|^2 \, ds = 1 \, N
\]

where \( \kappa_m^2 > 0 \), \( m = 1, 2, 3, \ldots \) are the eigenvalues. The potential \( \phi_0(\mathbf{r}) \) will be zero.

The solution \( \phi_m(\mathbf{r}) \) to the Dirichlet problem (10) generates the TM time-domain modal fields with the following components
\[ H^e_{\nu m} = 0 \]
\[
\kappa_m^{-1} \mathbf{H}^e_m = \left\langle -\partial_{\kappa_m e}, e_m(z, t) \right\rangle \left[ \mathbf{z} \times \mu_0^{-1} \kappa_m A^TM_m \nabla_\perp \phi_m(\mathbf{r}) \right]
\]
\[
\kappa_m^{-1} \mathbf{E}^e_m = \left\langle \partial_{\kappa_m e}, e_m(z, t) \right\rangle \left[ \varepsilon_0^{-1} \kappa_m A^TM_m \nabla_\perp \phi_m(\mathbf{r}) \right]
\]
\[
\kappa_m^{-1} \mathbf{E}^\nu_m = \left\langle e_m(z, t) \right\rangle \left[ \varepsilon_0^{-1} \kappa_m A^TM_m \phi_m(\mathbf{r}) \right]
\]

where \( \partial_{\kappa_m e} = (1/c \kappa_m) \partial / \partial t, \partial_{\kappa_m z} = (1/\kappa_m) \partial / \partial z \). The potential \( e_m(z, t) \) generates the modal amplitudes in Eq. (11) is the solution of the KGE as

\[
\left( \partial^2_{\kappa_m e} - \partial^2_{\kappa_m z} + 1 \right) e_m(z, t) = 0
\]

which is similar to Eq. (9) [5].

The factors selected by the square brackets [.] in Eq. (7) and (11) describe the modal field patterns in the waveguide cross section. Their physical dimensions are \( V m^{-1} \) and \( A m^{-1} \) for the electric and magnetic field components, respectively. The factors selected by the broken brackets \( \langle \rangle \) in Eq. (7) and (11) are dimensionless. Their physical sense is about the time-dependent modal amplitudes of appropriate modal field components.

The set of the TE and TM modes (as the vector functions of transverse coordinates) is complete due to the completeness of their generating potentials in the same energetic space. The completeness comes from Sturm-Liouville and Weyl theorem in functional analysis about the orthogonal detachments of Hilbert space \( L_2(S) \) [3]-[5]. This energetic space can be specified by an inner product as

\[
\left( X_1, X_2 \right) = \frac{1}{S} \int_S (\varepsilon_0 E_1 E_2 + \mu_0 H_1 H_2) dS < \infty
\]

where \( X_i = \text{col}(E_i, H_i), i = 1, 2, \ldots \), col. stands for "column". One can verify that \( \left( X^TE_m, X^TM_n \right) = 0 \) for any combinations of \( m \) and \( n \) with the values of 0,1,2,..., independently. Therefore, any pair of the TE and TM time-domain modes is orthogonal in the sense of inner product (13).

### 2.2. The time-dependent modal amplitude problem

The KGE in Eq. (9) for the TE modes and the KGE in Eq. (12) for the TM modes have the same structure. After introducing the scaled time \( \tau \) and scaled coordinate \( \xi \) as

\[
\tau = \nu_m c t \quad , \quad \xi = \nu_m z \quad \text{for TE modes}
\]
\[
\tau = \kappa_m c t \quad , \quad \xi = \kappa_m z \quad \text{for TM modes}
\]

the KGE in Eq. (9) and Eq. (12) can be written in the general form of

\[
\left( \partial^2_{\tau} - \partial^2_{\xi} + 1 \right) f(\xi, \tau) = 0
\]
where \( f(\xi, \tau) \) is either \( h_m(\xi, \tau) \) provided that \( \xi = \kappa_m z \) and \( \tau = \kappa_m ct \) or \( e_m(\xi, \tau) \) provided that \( \xi = \nu_m z \) and \( \tau = \nu_m ct \).

The KGE maintains its form under an action of a Poincare group within the framework of the group theory. In this aspect, Miller established eleven so called orbits of symmetry in terms of the group theory [13]. His results are crucial for development of the electromagnetic field theory in the time-domain.

On the basing of Miller’s idea, let us interpret solution to the KGE in Eq. (15) as a function with a new arguments, namely: \( f \equiv f(\xi, \tau) = f[u(\xi, \tau), v(\xi, \tau)] \). The “new” variables \((u, v)\) are unknown yet, but suppose that they are twice differentiable functions of the “old” variables \((\xi, \tau)\). Substitution of \( f[u(\xi, \tau), v(\xi, \tau)] \) as a formal solution to Eq. (15) yields a new form of this equation as

\[
\left[ \left( \frac{\partial u}{\partial \tau} \right)^2 - \left( \frac{\partial u}{\partial \xi} \right)^2 \right] \frac{\partial^2 f}{\partial u^2} + \left[ \left( \frac{\partial v}{\partial \tau} \right)^2 - \left( \frac{\partial v}{\partial \xi} \right)^2 \right] \frac{\partial^2 f}{\partial v^2} + \left[ \frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial \xi^2} \right] \frac{\partial f}{\partial u} \nonumber \\
+ \left[ \frac{\partial^2 v}{\partial \tau^2} - \frac{\partial^2 v}{\partial \xi^2} \right] \frac{\partial f}{\partial v} + 2 \left[ \frac{\partial u}{\partial \tau} \frac{\partial v}{\partial \tau} - \frac{\partial u}{\partial \xi} \frac{\partial v}{\partial \xi} \right] \frac{\partial^2 f}{\partial u \partial v} = f = 0
\]

where notice that the derivatives \( \partial_u \) and \( \partial_v \) act on the function \( f(u,v) \) under study. The various combinations of the derivatives by \( \xi \) and \( \tau \) of the functions \( u(\xi, \tau) \) and \( v(\xi, \tau) \) are appeared (unknown yet!) at the coefficients placed in square brackets. In order to solve Eq. (16), it is necessary to perform the following operations: a) Define the proper functions of \( u \) and \( v \). b) Express the coefficients (placed in square brackets) as the functions of \( u \) and \( v \). After this step, Eq. (16) becomes a PDE with variable coefficients depending on \( u \) and \( v \). c) Solve Eq. (16) via separation of the variables \( u \) and \( v \). This can be done if and only if the functions of \( u \) and \( v \) are specified properly. For this aim, Miller obtained eleven pairs of inverse functions, i.e., \( \xi(u,v) \) and \( \tau(u,v) \) [13]. As an example, the first pair is \( \xi = v \) and \( \tau = u \) where \( -\infty < u < \infty, \quad -\infty < v < \infty \). In this case, \( f(u,v) \) will be a product of the exponential functions and yields the classical time-harmonic waves.

Specially, we account the second pair as \( \tau = ucosh(v) \) and \( \xi = usinh(v) \) for the range of \( -\infty \leq u < \infty, \quad -\infty < v < \infty \). This yields a product of an exponential and Bessel functions. Inversion of these equations i.e. \( \tau = ucosh(v) \) and \( \xi = usinh(v) \) results in

\[
\xi(\xi, \tau) = \sqrt{\tau^2 - \xi^2}, \quad \tau(\xi, \tau) = \arctanh \left( \frac{\xi}{\tau} \right) \equiv \frac{1}{2} \ln \left( \frac{\tau + \xi}{\tau - \xi} \right).
\]

In this case, the calculation of the coefficients in Eq. (16) yields

\[
\left( \frac{\partial u}{\partial \tau} \right)^2 - \left( \frac{\partial u}{\partial \xi} \right)^2 = 1, \quad \left( \frac{\partial v}{\partial \tau} \right)^2 - \left( \frac{\partial v}{\partial \xi} \right)^2 = -\frac{1}{u^2}, \quad \frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial \xi^2} = \frac{1}{u^2}.
\]

The other coefficients are zero. Substitution of these coefficients to Eq. (16) simplifies it to the following form:
The Bernoulli’s product method suggests to apply factorization as \( f(u,v) = U(u)V(v) \) for the solution of Eq. (19). It results the following pair of ordinary differential equations

\[
U^{\alpha}(u) + \frac{1}{u} U'(u) + \left( 1 - \frac{\alpha^2}{u^2} \right) U(u) = 0, \quad V^{\alpha}(v) - \alpha^2 V(v) = 0
\]  

(20)

where \( \alpha \) is a constant for separation of the variables. The Bessel’s differential equation (first one) gives chance to choose this constant may be integer, fractional or even complex-valued numbers [13]. The second differential equation in Eq. (20) is an ordinary differential equation. Every equation has two linearly independent solutions. Combination of them yields the physically reasonable result in terms of the variables \((\xi, \tau)\) as

\[
f_a(\xi, \tau) = C_a \left( \frac{\tau - \xi}{\tau + \xi} \right)^{\frac{\alpha}{2}} J_{\alpha} \left( \sqrt{\tau^2 - \xi^2} \right) + D_a \left( \frac{\tau + \xi}{\tau - \xi} \right)^{\frac{\alpha}{2}} J_{\alpha} \left( \sqrt{\tau^2 - \xi^2} \right)
\]  

(21)

where arbitrary constant factors are omitted. It can be seen that the solution is symmetric with respect to the point \( \xi = 0 \). From this point, we can choose the solution for \( \xi > 0 \)

\[
f_a(\xi, \tau) = \left( \frac{\tau - \xi}{\tau + \xi} \right)^{\frac{\alpha}{2}} J_{\alpha} \left( \sqrt{\tau^2 - \xi^2} \right)
\]  

(22)

Physically, Eq. (22) is the time dependent modal amplitude of the longitudinal field component and corresponds to \( h_m(z,t) \) in Eq. (7) or \( e_m(z,t) \) in Eq. (11).

The analysis of the modal amplitudes of the \( TE \) and \( TM \) fields can be executed in parallel according to Eq. (7) and Eq. (11). The amplitudes of the transverse field components are presentable by the same formulas, namely [5],[6]

\[
A(\xi, \tau) = -\frac{\partial}{\partial \tau} f_a(\xi, \tau) \equiv \left\{ -\partial_{\nu_e} h_m \right\} \equiv \left\{ -\partial_{\nu_m} e_m \right\}
\]

\[
B(\xi, \tau) = \frac{\partial}{\partial \xi} f_a(\xi, \tau) \equiv \left\{ \partial_{\nu_e} h_m \right\} \equiv \left\{ \partial_{\nu_m} e_m \right\}
\]  

(23)

The modal amplitudes of the longitudinal components in Eq. (7) and (11) both are the solutions to the KGE (15). Then, the modal amplitudes of the transverse field components can be specified by Eq. (23). The direct differentiations of \( f_a(\xi, \tau) \) in accordance with these formulas result in
\[ A_j(\xi, \tau) = \frac{J_1(\sqrt{\tau^2 - \xi^2})}{\sqrt{\tau^2 - \xi^2}}, \quad B_\alpha(\xi, \tau) = \frac{J_1(\sqrt{\tau^2 - \xi^2})}{\sqrt{\tau^2 - \xi^2}} \]

\[ A_\alpha(\xi, \tau) = -\frac{\partial}{\partial \tau} f_\alpha = -\left(\frac{f_{\alpha-1} - f_{\alpha+1}}{2}\right), \quad B_\alpha(\xi, \tau) = -\frac{\partial}{\partial \xi} f_\alpha = -\left(\frac{f_{\alpha-1} + f_{\alpha+1}}{2}\right) \]

where the free parameter \( \alpha > 0 \).

3. General properties of the time-domain modes

3.1. Relativistic invariance of the time-domain modes

Maxwell’s equations with time derivative (1) can be adapted to axiomatics of the special theory of relativity. It can be shown that the proposed time-domain solution as the particular solutions to Eq. (1) correlates this theory properly as well [6].

Let the modal fields (7) and (11) correspond to an inertial reference frame \( F \) specified by the coordinates and time \((r, z, t)\). Introduce a new inertial reference frame, say \( F' \) with coordinates and time \((r', z', t')\) that executes motion only with the constant speed \( v \) along the axis \( Oz \). Correspondence between \((r, z, t)\) in \( F \) and \((r', z', t')\) in \( F' \) can be specified by a direct Lorentz transform, i.e.,

\[ r = r', \quad z = (z' + vt') \gamma, \quad t = (t' + vz'/c^2) \gamma \]

where \( \gamma = 1/\sqrt{1 - \beta^2}, \quad \beta = v/c \). The inverse transform can also be specified symmetrically as

\[ r = r', \quad z = (z - vt) \gamma, \quad t' = (t - vz/c^2) \gamma. \]

As far as \( r = r' \) in Eq. (25) and vice versa in Eq. (26), the Neumann problem (5) and the Dirichlet problem (10) have the same form in \( F \) and in \( F' \) both. Therefore, the potentials \( \psi_m(r) \) and \( \phi_m(r) \) specifying a modal basis are the same in \( F \) and in \( F' \) as well. It means that the modal basis is invariant under Lorentz transform.

It can also be shown that the KGE in Eq. (15) has also the same form in the reference frames \( F \) and in \( F' \) both. Let us interpret the solution \( f(\xi, \tau) \) to Eq. (15) as a function with the argument \( f(\xi', \tau') \) where “new” variables \((\xi', \tau')\) are specified as the functions of the “old” variables \((\xi, \tau)\) by inverse Lorentz transformation. Then, reiteration of the procedures in Eq. (15) yields

\[ \left(\partial_\tau^2 - \partial_\xi^2 + 1\right)f(\xi', \tau') = 0. \]

It is clear that the KGE in Eq. (27) for the reference frame \( F' \) has the same form as in Eq. (15) for the original reference frame \( F \) [6]. It means that the KGE is also invariant under Lorentz transformation. Therefore, the proposed time-domain solution is invariant under Lorentz transformation.
3. 2. The initial conditions for Klein-Gordon equation

The KGE (15) has to be supplemented with a pair of initial conditions. Physically, they specify the source signal for excitation. Suppose that such source is turned on at \( t = 0 \), however it does not act before. Then the initial conditions at \( \xi = 0 \Rightarrow z = 0 \) can be written as

\[
\left. f(\xi, \tau) \right|_{\xi=0} = \begin{cases} \phi(\tau), & \tau \geq 0 \Rightarrow t \geq 0 \\ 0, & \tau < 0 \Rightarrow t < 0 \end{cases}, \quad \left. \frac{\partial}{\partial \tau} f(\xi, \tau) \right|_{\xi=0} = \begin{cases} \phi(\tau), & \tau \geq 0 \Rightarrow t \geq 0 \\ 0, & \tau < 0 \Rightarrow t < 0 \end{cases}. (28)
\]

where \( \phi(\tau) \) and \( \tilde{\phi}(\tau) \) have to be known functions.

3. 3. The causality principle

The solution of the KGE have to be subjected for the physical requirements of the causality principle which can be interpreted in two ways: First, a weak causality condition states that all fields are zero before their sources are not turned on. In our case, this corresponds to \( \tau < 0 \) which relates to the initial condition. Second, a strong causality condition from the Einstein postulates that the electromagnetic field can not transfer energy more than the speed of the light \( c \) in the vacuum. In our case, this implies that the solution of the KGE should be zero beyond the distance \( \xi = \tau \) (i.e., \( z = c\tau \)) which corresponds to the wavefront of the electromagnetic wave. Thus, the solution of the KGE can be read physically as

\[
f(\xi, \tau) = \begin{cases} f(\xi, \tau) = 0, & \tau < 0 \\ f(\xi, \tau) \neq 0, & 0 \leq \xi \leq \tau \\ f(\xi, \tau) = 0, & \xi > \tau \end{cases}. \quad (29)
\]

4. Energy and surplus of energy for the time-domain modes

Let’s select a volume \( V \) located between two waveguide cross sections along the coordinates \( z \) and \( z + \delta z \) where \( z \) and \( \delta z > 0 \) are arbitrary. After applying the Poynting theorem to Maxwell’s equations for this volume with the limiting case of \( \delta z \to 0 \), it yields law of the conservation for any modal field energy as

\[
\frac{\partial}{\partial \xi} P_\xi(\xi, \tau) + \frac{\partial}{\partial \tau} W(\xi, \tau) = 0 \quad (30)
\]

where \( P_\xi(\xi, \tau) \) and \( W(\xi, \tau) \) are the \( z \)-component of the Poynting vector averaged over the waveguide cross section and averaged modal field energy stored in the same cross section, respectively. Eq. (30) known as “continuity equation” holds for every modal field in Eq. (7) and (11) individually. Physically, the continuity equation is a form of the conservation of the modal field energy law. \( P_\xi(\xi, \tau) \) and \( W(\xi, \tau) \) can be specified by \( A(\xi, \tau) \) and \( B(\xi, \tau) \) in Eq. (23) as in [7]

\[
P_\xi(\xi, \tau) = cA(\xi, \tau) \times B(\xi, \tau), \quad W(\xi, \tau) = \frac{A^2(\xi, \tau) + B^2(\xi, \tau) + f^2(\xi, \tau)}{2}. \quad (31)
\]
Mathematically, the \( P \) and \( W \) specify the \textit{global} properties of the time-domain modes in the energetic space of the solutions whereas the continuity equation (Eq. (30)) specifies their \textit{local} properties in this space. In this case, the \textit{energetic} field characteristics for all the time-domain modes can be represented as

\[
\begin{align*}
  dW_a(\xi, \tau) &= \frac{A_a^2(\xi, \tau) - B_a^2(\xi, \tau)}{2}, \quad W_a(\xi, \tau) = \frac{1}{2} \left( \frac{\tau - \xi}{\tau + \xi} \right) J_a \left( \sqrt{\tau^2 - \xi^2} \right) \\

\end{align*}
\]

where \( f_a(\xi, \tau) \) represents the solution of the KGE. In this work, the \textit{energetic} fields are specially treated for the \textit{semi-integer} numbers (\( \alpha = n + 1/2, \ n = 0, 1, 2, \ldots \)) as a case example. This corresponds to the spherical Bessel functions which can be represented in the direct form of trigonometric functions.

Fig. 1. Exchange of \( W_{1/2} \) (---) and \( dW_{1/2} \) (-----) along \( \xi \) at fixed \( \tau = 20 \).

Fig. 2. Exchange of \( dW_{3/2} \) (---) and \( d_{3/2} \) (-----) along \( \xi \) at fixed \( \tau = 30 \).

Fig. 3. Exchange of \( W_{1/2} \) (---) and \( dW_{1/2} \) (-----) along \( \tau \) at fixed \( \xi = 10 \).

Fig. 4. Exchange of \( W_{3/2} \) (---) and \( dW_{3/2} \) (-----) along \( \tau \) at fixed \( \xi = 10 \).
Fig. 5. Exchange of $W_{5/2}$ (---) and $dW_{5/2}$ (••••) along $\tau$ at fixed $\xi = 10$.

Fig. 6. Exchange of $W_0$ (——) and $dW_0$ (••••) along $\tau$ at fixed $\xi = 10$.

Probably, the physical interpretation of the graphical results given from Fig. 1 to Fig. 6 can be facilitated via the presentation of the same data recalculated for the following energetic quantities

$$W_0(\xi, \tau) = \frac{1}{2} f_0^2(\xi, \tau), \quad dW_0(\xi, \tau) = \frac{A_0^2(\xi, \tau) - B_0^2(\xi, \tau)}{2}. \quad (33)$$

According to that, the propagation of the transient signals generated by the potential $f_0(\xi, \tau)$ can be interpreted as an energetic transient wave process. The longitudinal component of a modal field with the stored energy $W_0$ transfers it to the transversal fields in a way which supports a surplus of energy $dW_0$ stored in the transverse modal field components. They participate in the energetic wave process of exchange by energy stored in the longitudinal and transverse field components.

The propagation of the steady-state time-harmonic modal field can also be interpreted in a similar way. Indeed, the function $f = \sin(\vartheta)$ (where $\vartheta = [\tau(\omega/\omega_m) - \xi \gamma_m]$, $\gamma_m = \sqrt{\omega^2 / \omega_m^2 - 1}$, $\omega$ is a frequency) is also a particular solution to the KGE in Eq. (15). It is the modal amplitude of the longitudinal three-component part of the modal field. The transverse component of this part has amplitude $B = -\gamma_m \cos \vartheta$. Another part of the modal field is transverse two-component fields with the amplitude $A = -\omega / \omega_m \cos \vartheta$. Therefore, the following relation can be stated as

$$W = \sin^2 \left[ \tau \left( \frac{\omega}{\omega_m} \right) - \xi \gamma_m \right], \quad dW = \cos^2 \left[ \tau \left( \frac{\omega}{\omega_m} \right) - \xi \gamma_m \right]. \quad (34)$$

where it is easy to see that $W + dW = 1$ for arbitrary time $\tau$ and coordinate $\xi$ [6].
5. Concluding remarks

In this study, the time-domain waveguide modes are expressed analytically by a method of Evolutionary Approach to Electromagnetics (EAE). A hollow waveguide is considered with the perfect electric conductor surfaces. Specially, the energy and surplus of the energy are investigated in details via the first kind Bessel functions of semi-integer order. Thus, the energetic wave process of exchange by energy stored in the longitudinal and transverse field components is introduced in the time-domain, directly. As the future works, the other possible solutions proposed from the Miller’s eleven cases will be considered for the solution of different problems such as partially filled lossless and lossy waveguides.

References