Application of Magneto-Quasi-Static Approximation in the Finite Difference Time Domain Method

Mehmet Burak Özakın and Serkan Aksoy
Electronics Engineering Department, Gebze Technical University, Gebze 41400, Turkey

Investigations about induction sensors, electromagnetic launchers, shields, transformers, and power line-induced currents address increased number of low-frequency research and industrial applications. In general, a magneto-quasi-static (MQS) approximation is considered for the solutions of low-frequency problems in electromagnetics. This approximation leads to a diffusion process when displacement currents are neglected. However, keeping the displacement currents, Maxwell’s equations are valid at low frequencies. In this manner, the finite difference time domain (FDTD) method must be modified by the slowing down propagation velocity at low-frequency regime. In this paper, important and crucial points of the MQS approximation and its application in the FDTD method are clarified in the sense of analytical and numerical aspects. A material scaling technique of dielectric permittivity for the QS FDTD application is analyzed within comprehensive investigations. Furthermore, a criterion for choosing a proper value of the scaling parameter will be revealed. Finally, effects of proper and improper values of the scaling parameter are presented with validated analytical and numerical results.

Index Terms—Finite difference time domain (FDTD) method, low-frequency induction, quasi-static (QS) approximation, scaling down propagation velocity.

I. INTRODUCTION

FINITE difference time domain (FDTD) is a well-adapted method for numerical solutions of complex electromagnetic problems. One of the important classes of such problems is low-frequency applications. Low-frequency problems, such as induction sensors, electromagnetic launchers, magnetic shields, transformers, and power line-induced currents, have very slow waveforms, in corresponding to electrically small targets. These problems can be solved either by a static assumption or using a quasi-static (QS) approximation. The QS approximation is more valuable, because low-frequency problems generally require the calculation of time domain-induced voltages. In this way, there are two fundamental routes as electro-QS (EQS) and magneto-QS (MQS) approximations.

Only MQS approximation is considered in this paper due to wide engineering applications of low-frequency sources. In the MQS approximation, displacement currents can be completely neglected. Thus, the problem is converted to a diffusion equation problem. This invalidates the electric charge continuity equation. It means that the transient behavior of charges cannot be calculated. An essential parameter in the low-frequency problem is current densities (eddy currents) induced in lossy materials. Therefore, a diffusion process is not suitable for eddy current problems. In addition, other weak points in this manner are: 1) decay of induced currents near wedges cannot be characterized by diffusion equation; 2) fields inside conductive medium obey Maxwell’s equations, not the diffusion equation; 3) induced currents are a result of Maxwell’s curl H equation, not the diffusion equation; and 4) finite velocity of wave propagation concept is also violated in this way [1].

Computational modeling plays a critical role for low-frequency problems due to complex wave interactions. Nonetheless, the solution of the diffusion equation by the explicit FDTD method may suffer from numerical difficulties, such as instabilities, very small unit time step, special treatment of material boundary conditions, and weakness of absorbing boundary conditions. Therefore, the direct solution of Maxwell’s equation is preferable in engineering problems. A classical FDTD method is one of the most widely used methods for such problems. However, it must be modified under the MQS approximation. In order to meet this requirement, the magneto-QS FDTD (QS-FDTD) method is proposed [1], [2]. In this method, keeping the displacement currents under the MQS condition, propagation velocity is principally slowed down by scaling up dielectric permittivity or magnetic permeability. This relaxes FDTD unit time step and makes the problem solvable in reasonable computational time and resources. Furthermore, the QS-FDTD method gives the full wave solution without ignoring coupling between electric and magnetic fields.

Up to date, the FDTD method has been employed for low-frequency problems. Some of the noticeable applications are induction sensors (metal detectors) [3], electromagnetic launchers [4], and power line-induced currents in the human body [5]. In fact, the FDTD method can be applied to the low-frequency problems with different views of the following:

1) velocity scaling technique [2];
2) field decoupling technique [5];
3) high definition technique [6];
4) frequency scaling technique [7].

In the field decoupling technique, electric and magnetic fields are completely decoupled. Therefore, this is not suitable for time domain problems. The high definition technique uses a peak detection method to reduce the computational time. Thus, it is not appropriate for the calculation of early time responses. The frequency scaling technique is based on the
solution of problems at relatively high frequencies. It is not directly related to the FDTD method.

In this paper, a material scaling technique in the FDTD method based on the MQS approximation (called the QS-FDTD method) is analyzed in a detailed manner. In this technique, velocity is scaled down by scaling up dielectric permittivity. Criteria for choosing a proper value of the scaling parameter are revealed in the content of physical considerations.

An outline of this paper is as follows. In Section II, the fundamentals of the QS and the MQS approximations are given. In Section III, the application of the MQS approximation in the FDTD method based on a proper choice of the scaling factor is analyzed. Different scaling methods are also discussed. In Section IV, ideas given in Sections II and III are compared with those of analytical and classical FDTD results supplied with a numerical example. The numerical results are discussed. In Section V, ideas given in Sections II and III are analyzed. Different scaling methods are also given. In Section III, the application of the MQS approximation is used through this paper.

In this paper, we focus on the MQS approximation because of wide engineering applications of low-frequency sources. The MQS approximation means that displacement currents are kept under the MQS condition. This gives rise to slow down propagation velocity by the material scaling technique. This makes the problem solvable with the FDTD method by relaxing its unit time step.

Now, let us analyze how fundamental parameters are affected by the MQS approximation.

1) Propagation Constant: The propagation constant becomes a complex number in the lossy medium as

$$k = k_{\text{real}} + jk_{\text{imag}} = \sqrt{\omega^2 \varepsilon \mu + j \omega \sigma \mu}$$

where the effect of the loss and the MQS condition must be considered together over real ($k_{\text{real}}$) and imaginary parts ($k_{\text{imag}}$) as

$$k_{\text{real}} = \sqrt{\frac{\varepsilon_0 \mu_0}{k_0} \sqrt{\varepsilon_r \mu_r} \left( \frac{1}{2} \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon_0 \varepsilon_r} \right)^2} + 1 \right)}$$

and

$$k_{\text{imag}} = \sqrt{\frac{\varepsilon_0 \mu_0}{k_0} \sqrt{\varepsilon_r \mu_r} \left( \frac{1}{2} \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon_0 \varepsilon_r} \right)^2} - 1 \right)}$$

where $k_{\text{real}}$ models wave propagation and $\sqrt{\alpha_k}$ times bigger than $k_0$. $k_{\text{imag}}$ explains a material loss mechanism and also less than $k_0$. The term $\sigma / \omega$ makes $k_{\text{real}}$ and $k_{\text{imag}}$ less $\omega$ and lesser $\varepsilon$-dependent. They are strongly dependent on $\mu$.

The coefficient $\alpha_k$ is a very large number because of the dominant term $\sigma / \omega c$. Therefore, the propagation velocity $c$ in the lossy medium reduces as

$$c = \frac{\omega}{k_{\text{real}}} = \frac{\omega}{\sqrt{\alpha_k} k_0} = \frac{1}{\sqrt{\alpha_k} c_0}$$

where $c \ll c_0$ because $k_{\text{real}} \gg k_0$. This fact implies that the propagation velocity in the conductive media slows down and
is much smaller than $c_0$. Although it is analytically clear, its numerical FDTD counterpart needs more investigations.

Specially, if and when $k_{\text{real}}$ and $k_{\text{imag}}$ are equal, the displacement currents are completely neglected. They can be approximated as

$$k \cong \sqrt{j\omega \mu \sigma} \Rightarrow k_{\text{real}} = k_{\text{imag}} \cong \frac{\omega \mu \sigma}{2}$$

where this way directly results to the diffusion equation.

2) Skin Depth: The skin depth ($\delta$) is formulated as

$$\delta = \frac{1}{k_{\text{real}}} = \frac{1}{\sqrt{\varepsilon \sigma}}$$

$\delta$ can be approximated under the MQS condition as

$$k_{\text{real}} \cong \frac{\omega \mu \sigma}{2} \Rightarrow \delta = \frac{1}{k_{\text{real}}} \cong \frac{2}{\omega \sigma \mu}$$

3) Impedance: The wave impedance in the lossy medium is

$$Z = \frac{E}{H} = \frac{\mu}{\varepsilon + j\sigma}$$

where it will be affected by material scaling. Therefore, rescaling must be considered. The impedance can also be approximated if the displacement currents are neglected as

$$Z = \frac{E}{H} = \frac{\mu \omega}{\varepsilon \sigma}.$$

4) Critical Model Time Parameters and Continuity Equation: In order for better understanding the low-frequency phenomenon, three fundamental parameters of interest have to be considered in the MQS problems: relaxation time $\tau_e$, transmit time $\tau_l$, and magnetic diffusion time $\tau_m$. Respectively, they are independently extracted from continuity, Maxwell’s, and diffusion equations [9].

The relaxation time (or conductive dielectric diffusion time) $\tau_e = \varepsilon / \sigma$ plays an important role for charge transition from transient regime to steady state. $\tau_e$ originates from the solution of the electric charge continuity equation. In lossy objects, the transient charges survive only during the relaxation time. Negligence of the displacement currents, leading to the diffusion equation, violates the continuity equation. Therefore, the interpretation of the transient-induced currents in terms of transient charge transportation is not possible. Hence, the diffusion process is not suitable for the transient charge problems. However, keeping the displacement currents and slowing down the propagation velocity, the continuity equation remains valid. Therefore, transient charges can be modeled in this way.

The transmit time $\tau_l = L / c = L \sqrt{\varepsilon \mu}$ models the electromagnetic wave propagation, where $L$ is the characteristic dimension of the object.

The magnetic diffusion time $\tau_m = \sigma \mu L^2$ can be treated as an indication of when the numerical simulation can be stopped. The computation time must be large enough to decay transients sufficiently.

Principally, the following condition for the MQS approximation must be satisfied for conductive media [9]:

$$\tau_e < \tau_l < \tau_m$$

where $\tau_l$ is replaced between $\tau_e$ and $\tau_m$ because it is a geometric mean of them.

In order to preserve the physics of the MQS phenomenon, the above sequence should not be violated by the material scaling technique in the numerical solution.

III. MAGNETO-QUASI-STATIC APPROXIMATION IN THE FDTD METHOD

Electrically, small problems at low frequencies are not directly solvable by the classical FDTD method, because the spatial unit step ($\Delta x$) has to be extremely small for accurate modeling of object geometry in which the object size is too small according to the wavelength. This requires extremely small unit time step ($\Delta t$) due to the stability criterion of the classical FDTD method. Therefore, computational time will be precluded by a huge number of time steps. This makes the problem unsolvable in reasonable computational time by today’s computer technology. Furthermore, accumulative numerical dispersion errors and long-time instabilities have to be considered for long iteration numbers. In fact, the extremely small $\Delta t$ is also not physically required.

The QS-FDTD method has been proposed to circumvent these difficulties. It is based on reducing the propagation velocity whenever the QS ($kr \ll 1$) and the MQS approximations ($\omega \varepsilon \ll 1$) are satisfied. Thus, the unit time step of the classical FDTD method is relaxed in the manner of the stability criterion by slowing down the propagation velocity. This gives chance to solve the low-frequency problems with the FDTD method. As discussed in Section II, slowing down the light velocity can be performed by the material scaling technique. This is principally possible by scaling up the parameters $\varepsilon$, $\mu$, $\sigma$, and $\omega$. However, the FDTD stability condition for lossy medium limits the case [11] and only scaling down the dielectric permittivity ($\varepsilon$) becomes feasible. Limitations for these parameters are discussed in the next parts of this section.

It seems that Luebbers et al. [8] first brought up the idea of $\varepsilon$-scaling, and then Holland applied this idea to a magnetic diffusion problem [2]. In the QS-FDTD method, the displacement currents are kept. This preserves the finite propagation velocity concept and yields to the validity of the charge conservation law.

In order to understand the numerical behavior of the FDTD method in the lossy medium, it is necessary to extract the numerical wave number ($k^N$) as

$$k^N = \omega \sqrt{\varepsilon \mu} \cong \sqrt{\left(\frac{\varepsilon}{\sigma} + j\frac{\sigma \Delta t}{2 \tan(\omega \Delta t/2)}\right) \mu}$$

$^1$The relaxation is a local phenomenon, but the diffusion is a global phenomenon. $\tau_e$ and $\tau_l$ lose their meaning for the diffusion equation due to the violation of the continuity equation and the finite propagation velocity concept. On that account, only $\tau_m$ remains a concern in the diffusion equation.
where \( \varepsilon_N \) is the numerical FDTD dielectric permittivity [11]. Let us calculate \( k_{\text{real}}^N \) and \( k_{\text{imag}}^N \) for analyzing numerical wave propagation

\[
k_{\text{real}}^N = \text{Re}(k^N) = \text{Re}(\omega \sqrt{\varepsilon_N \mu})
\]

\[
=k_0 \sqrt{\varepsilon_r \mu_r} \left( \frac{1}{2} \sqrt{1 + \frac{(\varepsilon_0 \varepsilon_r \mu_0 \mu_r)}{\varepsilon_0 \mu_0}} \right) \frac{\omega \Delta t / 2}{\tan(\omega \Delta t / 2)} + 1 \right)^{1/2}
\]

\[
k_{\text{imag}}^N = \text{Imag}(k^N) = \text{Imag}(\omega \sqrt{\varepsilon_N \mu})
\]

\[
=k_0 \sqrt{\varepsilon_r \mu_r} \left( \frac{1}{2} \sqrt{1 + \frac{(\varepsilon_0 \varepsilon_r \mu_0 \mu_r)}{\varepsilon_0 \mu_0}} \right) \frac{\omega \Delta t / 2}{\tan(\omega \Delta t / 2)} - 1 \right)^{1/2}
\]

where \( \tan(\omega \Delta t / 2) \approx \omega \Delta t / 2 \) for small argument approximation. Then, \( k_{\text{real}}^N \) and \( k_{\text{imag}}^N \) become

\[
k_{\text{real}}^N \approx k_0 \sqrt{\varepsilon_r \mu_r} \left( \frac{1}{2} \sqrt{1 + \frac{(\varepsilon_0 \varepsilon_r \mu_0 \mu_r)}{\varepsilon_0 \mu_0}} + 1 \right) \Rightarrow k_{\text{real}}^N = k_{\text{real}}
\]

\[
k_{\text{imag}}^N \approx k_0 \sqrt{\varepsilon_r \mu_r} \left( \frac{1}{2} \sqrt{1 + \frac{(\varepsilon_0 \varepsilon_r \mu_0 \mu_r)}{\varepsilon_0 \mu_0}} - 1 \right) \Rightarrow k_{\text{imag}}^N = k_{\text{imag}}
\]

where \( k_{\text{real}}^N \) and \( k_{\text{imag}}^N \) are equal to the analytical counterparts leading to the dispersionless FDTD solution. They are also weakly dependent on \( \varepsilon \).

In the numerical solution, the propagation velocity inherently slows down because \( k^N > k_0 \) due to the dominant term \( \sigma / \omega \varepsilon_0 \varepsilon_r \). Although it slows down, we cannot see its effect to relax \( \Delta t \) in the classical FDTD method because the stability criterion depends on \( \lambda_0 \) and \( c_0 \), not reduced \( \lambda \) and \( c \). Therefore, an artificial dimensionless parameter must be introduced in the material scaling technique. This reality settles a base for the QS-FDTD method.

In order to preserve the physics of the low-frequency phenomenon, a critical step in the QS-FDTD method is to find a proper (correct) value of the scaling factor. This problem has already been discussed in the literature [10]. The improper (incorrect) values will result in wrong solutions. Furthermore, effects of the material scaling over physical and numerical parameters need comprehensive investigations. Therefore, the QS-FDTD method requires careful treatment before its application.

The scaling affects following parameters and conditions:

1) the QS and MQS conditions;
2) parameters of \( r_\tau, \tau_r \), and \( n_\tau \);
3) parameters of \( \delta, Z \), and rescaling;
4) numerical dispersion and stability.

Before investigations about effects of the \( \varepsilon \)-scaling, let us first analyze general issues, such as geometric resolution and Nyquist scaling limitation.

Geometric Resolution: A key parameter in the material scaling technique is geometric resolution. At the beginning of the problem, it must be defined for desired geometric model accuracy. Then, analyzing the QS and the MQS approximations, a scaling parameter \( (\alpha) \) must be found in order to preserve the physics of the wave phenomenon.

Let us assume that the propagation velocity is slowed down by a factor of \( \sqrt{\alpha} \) as

\[
c_{\text{QS}} = \frac{1}{\sqrt{\alpha}} c_0 \Rightarrow \lambda_{\text{QS}} = \frac{1}{\sqrt{\alpha}} \lambda_0
\]

where \( c_{\text{QS}} \) and \( c_0 \) are the scaled (QS) and the unscaled (free-space) propagation velocities, respectively. \( \lambda_{\text{QS}} \) and \( \lambda_0 \) are the scaled (QS) and the unscaled (free-space) wavelengths, respectively. Clearly, slowing down the propagation velocity decreases the wavelength \( (\lambda_{\text{QS}} < \lambda_0) \). This improves geometric resolution. \( \sqrt{\alpha} \) is related to the desired geometric model resolution. In order to optimize this problem, it will be necessary to arrange the number of scaled calculation points \( N_{\text{QS}} \), which is a function of unscaled calculation points \( N \). These directly affect the computational requirements of the FDTD method.

Let us think a 1-D problem for the sake of simplicity. For the desired (predefined) geometrical resolution, first, it is necessary to define the value of \( \Delta x \)

\[
\Delta x = \frac{L}{N_L}
\]

where \( L \) is the largest size of the object and \( N_L \) is the desired number of the calculation points. After clarifying the value of \( \Delta x \) by this way, a link (geometric resolution equality) must be established between \( \Delta x \) and the scaled spatial unit step size \( \Delta x_{\text{QS}} \) as

\[
\Delta x = \Delta x_{\text{QS}} \Rightarrow \frac{\lambda_0}{N_x} = \frac{\lambda_{\text{QS}}}{N_{\text{QS}}}
\]

where \( N_x \) is the unscaled number of the FDTD calculation points. It is recommended to take \( N_{\text{QS}} \) at least 10 for the acceptable interaction of the waves with objects. Since the value of \( \alpha \) is not known from the beginning, a value of \( \lambda_{\text{QS}} \) is not clear yet. Therefore, \( N_{\text{QS}} \) cannot be found in this step. If the scaling factor \( \alpha \) is found from different criteria, then \( N_{\text{QS}} \) can be calculated as

\[
\frac{\lambda_0}{N_x} = \frac{\lambda_{\text{QS}}}{N_{\text{QS}}} \Rightarrow N_{\text{QS}} = \frac{N_x}{\sqrt{\alpha}}.\]  

The value of \( N_{\text{QS}} \) should be linked to the numerical algorithm.

Nyquist Limitation: Nyquist limitation is an important point in the QS-FDTD method. It is known that the classical \( \Delta t \) must be smaller than Nyquist unit time step \( \Delta t_{\text{N}} \) as

\[
\Delta t \leq \Delta t_{\text{N}} = \frac{1}{2 f_{\text{max}}}
\]

where \( f_{\text{max}} \) is the maximum operational frequency. The scaling relaxes \( \Delta t \) as \( \Delta t_{\text{QS}} = \sqrt{\alpha} \Delta t \), but the Nyquist criterion must be satisfied again as

\[
\Delta t_{\text{QS}} = \sqrt{\alpha} \Delta t \leq \Delta t_{\text{N}} = \frac{1}{2 f_{\text{max}}} \Rightarrow a_N \leq \left( \frac{1}{2 f_{\text{max}} \Delta t} \right)^2
\]

where the Maxwell equation-based \( a_k \) can violate above limitation and smaller one must be taken.

Other Considerations: One of the requirements in the QS-FDTD method is to apply the material scaling technique through whole problem space. In the problem space, the conductive objects can be located in free space, dielectric,
or conductive media. However, in order for global application of the QS-FDTD method, the MQS condition has to be satisfied not only for the conductive object, but also for surrounded medium. Therefore, it is necessary to add small conductivities for the region of the lossless medium in order to satisfy the MQS approximation, everywhere.

Now, $\varepsilon$-scaling, $\mu$-scaling, $\sigma/\varepsilon$-scaling, $\sigma$-scaling, and $\omega$-scaling will be analyzed in detail. Unconditionally, stable methods are also briefly discussed.

A. Dielectric Permittivity ($\varepsilon$) Scaling

The $\varepsilon$-scaling is based on scaling up the dielectric permittivity by a factor of $\alpha$ as $\varepsilon^{\text{QS}} = \alpha \varepsilon$ for slowing down the propagation velocity. This relaxes $\Delta t$ as $\Delta t^{\text{QS}} = \sqrt{\alpha} \Delta t$. Effects of the $\varepsilon$-scaling over the parameters are investigated below.

1) QS Condition and $k$: The $\varepsilon$-scaling does not strongly affect the QS condition and $k$, since $k_{\text{real}}$ is not strongly dependent on $\varepsilon$. Therefore, the QS range is not affected so much.

2) MQS Condition: Unlike the QS condition, MQS condition is directly affected by the $\varepsilon$-scaling. In order to preserve the MQS approximation, first restriction appears as $a = a_{\text{MQS}}$

\[
2\pi f a_{\text{MQS}} \varepsilon \ll \sigma \Rightarrow a_{\text{MQS}} \ll \frac{\sigma}{2\pi f \varepsilon}.
\]

3) $\tau_e$, $\tau_t$, and $\tau_m$: In order to properly model the physics of the MQS phenomenon in the QS-FDTD method, the relaxed unit time step $\Delta t^{\text{QS}}$ has to obey a condition of

\[
\Delta t^{\text{QS}} < \tau_e^{\text{QS}} < \tau_t^{\text{QS}} < \tau_m.
\]

This relation shows that the scaling factor $\alpha$ cannot be chosen as large as desired by using the relation of $\Delta t^{\text{QS}} = \sqrt{\alpha} \Delta t$. In this step, three cases must be separately analyzed in order to reveal the limitations of the scaling parameter $\alpha$. $\tau_e$ and $\tau_t$ are affected by the $\varepsilon$-scaling, but $\tau_m$ is not.

The first case

\[
\Delta t^{\text{QS}} = \sqrt{\alpha} \Delta t < \tau_e^{\text{QS}} = \alpha \tau_e \Rightarrow \alpha > \left( \frac{\Delta t}{\tau_e} \right)^2
\]

is affected by the $\varepsilon$-scaling by an order of $\alpha$. Here, an important point is very high conductive objects. The larger $\sigma$ means the smaller $\tau_e^{\text{QS}}$. This gives a severe restriction for the classes of conductive objects, which can be investigated by the QS-FDTD method. Therefore, if one wants to calculate the transient charge behavior, the QS-FDTD method for very high conductive objects becomes computationally unfeasible (similar to the classical FDTD method), since extremely small $\Delta t^{\text{QS}}$ requires impracticable computational time.

In the second case, relation $\tau_e^{\text{QS}} < \tau_t^{\text{QS}}$ must be kept as

\[
\tau_e^{\text{QS}} = \alpha \tau_e \Rightarrow \alpha = \alpha_t < \frac{\tau_m}{\tau_e}
\]

and, third, relation $\tau_t^{\text{QS}} < \tau_m$ must also to be kept

\[
\tau_t^{\text{QS}} < \tau_m \Rightarrow \sqrt{\alpha} \tau_t \tau_m < \tau_m \Rightarrow \alpha = \alpha_t < \frac{\tau_m}{\tau_t}
\]

where these conditions give the same restriction.

At first glance, one can conclude that larger scaling parameter is preferable, because it further relaxes $\Delta t$. However, according to above restrictions, this is not the case, since the physics of the MQS approximation has to be preserved in the QS-FDTD numerical solution.

4) $\delta$, $Z$, and Rescaling: The skin depth $\delta$ can be extracted as $\delta = 1/k_{\text{real}} \geq (2/\omega \sigma \mu)^{1/2}$. Hence, it is not strongly affected by the $\varepsilon$-scaling.

The impedance is affected by the $\varepsilon$-scaling as

\[
Z^{\text{QS}} = \frac{|E|^{\text{QS}}}{|H|^{\text{QS}}} = \sqrt{\frac{\mu}{\alpha \varepsilon + j \sigma^2}}; \quad Z = \sqrt{\frac{\mu}{\varepsilon + j \sigma^2}}
\]

where $Z^{\text{QS}}$ and $Z$ are the scaled and unscaled impedances, respectively. Under the MQS approximation, they can be modified as ($Z_0 = (\mu/\varepsilon)^{1/2}$)

\[
Z = \sqrt{\frac{\mu}{\varepsilon}} \sqrt{\frac{1}{1 + j \frac{\sigma}{\omega \varepsilon}}} \approx Z_0 \sqrt{\frac{1}{1 + j \frac{\sigma}{\omega \varepsilon}}}
\]

and

\[
Z^{\text{QS}} = Z_0 \sqrt{\frac{1}{1 + j \frac{\sigma}{\omega \varepsilon}}} \approx Z_0 \sqrt{1 + \frac{1}{1 + j \frac{\sigma}{\omega \varepsilon}}} = \frac{Z}{a'}
\]

where $a'$ can be defined as a rescaling factor between the scaled and the unscaled impedances. For amplitude calculations$^2$

\[
|a'| = \left| 1 - j \frac{\sigma}{\omega \varepsilon} \right| = \sqrt{1 + \left( \frac{1}{2} \frac{\sigma}{\omega \varepsilon} \right)^2}.
\]

Here, $a'$ is a problem-dependent parameter. If the source is of an electric field source, the magnetic field is scaled up by $|a'|$. Therefore, it must be rescaled down by a factor of $|a'|$. If the source is of magnetic field source, the electric field is scaled down by $|a'|$. Therefore, it must be rescaled up by a factor of $|a'|$. Here, it is also essential to mention near field region, since the low-frequency problems are generally in the near field region, where the impedance depends upon the type of excitation source and distance between transmitter and receiver. Therefore, the near field impedance may give better insight than the far field impedance.

5) Numerical Stability and Dispersion: In order to investigate the numerical behavior of the QS-FDTD method, let us perform the numerical analysis of the method in the lossy medium.

First, we start with a stability analysis. According to Pereda et al. [11], the FDTD stability criterion in the

$^2$The phase of the impedance will also be affected because the impedance is the complex number.
lossy medium has the same stability limit as the classical FDTD method. Hence, the QS-FDTD stability criterion is
\[
\frac{c_{QS} \Delta x_{QS}}{\Delta x} \leq 1 \Rightarrow \Delta x_{QS} \leq \frac{\Delta x}{c_{QS}} = \sqrt{\alpha} \frac{\Delta x}{c_0} \leq \sqrt{\alpha} \Delta t
\]
where the geometric resolution equality \( \Delta x = \Delta x_{QS} \) must be kept as discussed before. In fact, the relaxation of \( \Delta t \) with a degree of \( \sqrt{\alpha} \) comes from the stability condition due to propagation velocity downscaling.

Now, let us consider a dispersion analysis in the lossy medium. Using the QS-FDTD numerical dielectric permittivity \( \varepsilon_{N} \) [11], real and imaginary parts of the numerical wave number can be found as
\[
k_{\text{real}}^{N_{c}} = k_{0} \sqrt{\varepsilon_{r}} \mu_{c} \left( \frac{1}{2} \left[ \frac{1}{\sqrt{\varepsilon_{r}} \mu_{c}} \frac{2}{\omega \sqrt{\alpha} \mu_{c} \mu_{r}} \left( \frac{\omega \sqrt{\alpha} \Delta t / 2}{\tan(\omega \sqrt{\alpha} \Delta t / 2)} \right)^{2} + 1 \right] \right)^{1/2}
\]
\[
k_{\text{imag}}^{N_{c}} = k_{0} \sqrt{\varepsilon_{r}} \mu_{c} \left( \frac{1}{2} \left[ \frac{1}{\sqrt{\varepsilon_{r}} \mu_{c}} \frac{2}{\omega \sqrt{\alpha} \mu_{c} \mu_{r}} \left( \frac{\omega \sqrt{\alpha} \Delta t / 2}{\tan(\omega \sqrt{\alpha} \Delta t / 2)} \right)^{2} - 1 \right] \right)^{1/2}
\]
where if \( \omega \sqrt{\alpha} \Delta t / 2 \) goes to zero, term \( (\omega \sqrt{\alpha} \Delta t / 2) / \tan(\omega \sqrt{\alpha} \Delta t / 2) \) goes to one. Therefore, the \( \varepsilon \)-scaling cannot strongly affect the numerical solution because \( k_{\text{real}}^{N_{c}} \) and \( k_{\text{imag}}^{N_{c}} \) are also weakly dependent on \( \alpha \). Therefore, the \( \varepsilon \)-scaling reduces right-hand side of the equation with an order of \( \alpha \) as \( \nabla \cdot E(r, t) = \rho / \varepsilon \). Therefore, additional rescaling may become inevitable in the manner of continuity equation.

In order to explain steps for the \( \varepsilon \)-scaling technique, a flowchart is given in Fig. 1. According to this, the transient charge calculation is the most important step. If one wants to calculate them, limitation becomes more restrictive for the scaling parameter, yet it is generally inapplicable for good conductors due to very small \( \alpha \Delta t \).

All given considerations above are critical to choosing the proper value of the scaling parameter \( \alpha \) in the QS-FDTD method. Consequently, a chosen value has to satisfy all the requirements. Otherwise, the physics of the MQS phenomenon will be destructed in the solution. Therefore, the numerical results will be questionable and lead to wrong results.

Table I shows how the model parameters are affected by the \( \varepsilon \)-scaling technique.

### Table I: Effect of the \( \varepsilon \)-Scaling on the Model Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>The ( \varepsilon )-scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = \sqrt{\varepsilon \mu + j \omega \mu_{0} \sigma} )</td>
<td>Weakly affected</td>
</tr>
<tr>
<td>( Z = \mu / (\varepsilon + j \sigma / \omega) )</td>
<td>Weakly affected</td>
</tr>
<tr>
<td>( \varepsilon = 1 / \sqrt{\mu} )</td>
<td>Reduced</td>
</tr>
<tr>
<td>( \delta = 1 / k_{\text{real}} )</td>
<td>Weakly affected</td>
</tr>
<tr>
<td>( \tau_{e} = \varepsilon / \sigma )</td>
<td>Increased</td>
</tr>
<tr>
<td>( \tau_{t} = L / \sqrt{\mu} )</td>
<td>Increased</td>
</tr>
<tr>
<td>( \tau_{m} = \mu_{0} L^{2} )</td>
<td>Not affected</td>
</tr>
<tr>
<td>( \nabla \cdot E(r, t) = \rho / \varepsilon )</td>
<td>Affected</td>
</tr>
</tbody>
</table>

the MQS approximation. This possibility was mentioned in Holland’s works [1], [2]. However, the \( \mu \)-scaling is not feasible in the QS-FDTD solutions. To understand reason for this, let us extract \( k_{\text{real}}^{N_{c}} \) and \( k_{\text{imag}}^{N_{c}} \) as
\[
k_{\text{real}}^{N_{c}} = k_{0} \sqrt{\varepsilon_{r}} \alpha_{\mu} \left( \frac{1}{2} \left[ \frac{1}{\sqrt{\varepsilon_{r}} \alpha_{\mu}} \frac{2}{\omega \sqrt{\alpha} \mu_{c} \mu_{r}} \left( \frac{\omega \sqrt{\alpha} \Delta t / 2}{\tan(\omega \sqrt{\alpha} \Delta t / 2)} \right)^{2} + 1 \right] \right)^{1/2}
\]
\[
k_{\text{imag}}^{N_{c}} = k_{0} \sqrt{\varepsilon_{r}} \alpha_{\mu} \left( \frac{1}{2} \left[ \frac{1}{\sqrt{\varepsilon_{r}} \alpha_{\mu}} \frac{2}{\omega \sqrt{\alpha} \mu_{c} \mu_{r}} \left( \frac{\omega \sqrt{\alpha} \Delta t / 2}{\tan(\omega \sqrt{\alpha} \Delta t / 2)} \right)^{2} - 1 \right] \right)^{1/2}
\]
where both terms are strongly affected by the \( \mu \)-scaling. This leads to solve a problem with smaller-scaled wavelength and very high numerical loss. It means that the numerical solution in the sense of the \( \mu \)-scaled wavelength and the \( \mu \)-scaled loss will be very far from the analytical solution. Therefore, the \( \mu \)-scaling is not feasible.

### B. Other Scalings

1) Magnetic Permeability (\( \mu \)) Scaling: It seems that the \( \mu \)-scaling is suitable for low-frequency problems. It is based on scaling up the magnetic permeability by a factor of \( \alpha \) as \( \mu^{QS} = \alpha \mu \) for slowing down the propagation velocity under

![Fig. 1. Steps of the \( \varepsilon \)-scaling technique.](image-url)
2) Frequency ($\omega$) Scaling: The $\omega$-scaling seems to be effective for the $\Delta t$ relaxation. In fact, it is not useful. According to the stability condition ($\Delta t \leq \Delta x / c = 1 / N f$ where $\Delta x = \lambda / N$ and $c = \lambda / f$), $\omega$ can be scaled down to relax $\Delta t$. However, this is not good, since the lower $f$ affects the predefined geometric resolution due to the relation of

$$\Delta x = \Delta x^{QS} \Rightarrow \frac{\lambda}{N} = \frac{\lambda^{QS}}{N^{QS}}$$

where there is a problematic issue, since the lower $f$ causes a larger $\lambda^{QS} = c^{QS} / f$. In order to keep the predefined geometric resolution ($\Delta x = \Delta x^{QS}$), $N^{QS}$ must be increased by the same ratio of $\lambda / \lambda^{QS}$. This yields more computational memory and time. Furthermore, a considered low-frequency problem will be solved at lower frequencies, not preferable in the sense of the physics of the problem.

Here, it is worthy to note that there is no relation between the mentioned $\omega$-scaling and the frequency scaling discussed in the literature [7]. They have different routes.

3) Conductivity ($\sigma$) Scaling: It seems that the $\sigma$-scaling is also useful, since it produces bigger $k_{eqal}$ and smaller $c$. However, this is also problematic for the QS-FDTD solutions because of the stability criterion. According to the stability criterion, in order to relax $\Delta t$, principally, we can only scale up $\varepsilon$, $\mu$, and/or $\omega$ except for $\sigma$, which is not effective for $\Delta t$ relaxation.

Alternatively, the $\sigma$-scaling and/or the $\omega$-scaling are considered as another manner in the literature. Liu and Crozier [12] put forward the asymmetry problem of weak coupling due to a very big ratio between the displacement currents and the conduction currents for a gradient-induced eddy current problem.

They claimed that Maxwell’s equations lose their symmetry for a ratio $\sigma / \omega \varepsilon = 10^{16}$. In order to mitigate the asymmetry problem, they scale down $\sigma$ and scale up $\omega$ or scale down $\sigma$ and scale up $\mu$ in order to keep an unchanged propagation constant. However, they used $k = (j \omega \varepsilon \mu)^{1/2}$ of the diffusion equation instead of $k = (\omega^{2} \varepsilon^{2} \mu + j \omega \sigma \mu)^{1/2}$. Furthermore, they did not compare the numerical results of asymmetry and mitigated asymmetry problems [12].

C. Unconditionally Stable FDTD Methods

An alternative idea to relax $\Delta t$ is to use implicit algorithms, such as the Crank–Nicolson method and the alternating direction implicit method [14]. Although they are unconditionally stable, the MQS restriction has to be satisfied again in order to preserve the physics of the MQS phenomenon. Therefore, it is impossible to choose $\Delta t$ as large as desired in these methods. However, the implicit algorithms can be useful up to above given $\Delta t$ limit. This is favorable, since none of the parameters $\tau_{r}$, $\tau_{f}$, $\tau_{m}$ and $\delta$, $Z$, $k$ are modified (not scaled) in the numerical calculations. But, its unit time will not be competitive to the QS-FDTD method, because their numerical errors become larger for large $\Delta t$. In addition, the Nyquist criterion and the good conductor limitation are also prohibitive for these methods.

IV. Numerical Example and Validation

A low-frequency problem in an infinitely long two-layered conductive and non-permeable medium is considered in 1-D Cartesian coordinate. This example is chosen, because it has an analytical solution for comparisons, free from stair case and Mur’s absorbing boundary condition errors. A monochromatic point source with a frequency of 10 Hz (the free-space wavelength is $\lambda_{0} = 30 \times 10^{6}$ m) is excited in the first layer. In order to retain an electrically small concept, the unit spatial step is taken very small as $\Delta x^{QS} = \Delta x = \lambda_{0} / 10^{7} = 3$ m ($N = 10^{7}$). The corresponding classical (not scaled) FDTD unit time step is $\Delta t \leq \Delta x / c_{0} = 10^{-8}$ s. The geometry of the problem is shown in Fig. 2. The full length of the problem space ($2L_{x}$) is chosen as equal to the wavelength of the first medium.

The classical FDTD solution is problematic for this problem, because the extremely small $\Delta t$ requires enormous computational time. Therefore, the QS-FDTD method must be used for the solution. In order to relax $\Delta t$, the material ($\varepsilon$) scaling technique is applied, because the MQS condition in the original (not scaled) problem is well satisfied. As explained in Section III, $\varepsilon$ is scaled up.

The material parameters of the media are given in Table II. According to the QS-FDTD method, first, we have to define the value of the geometric resolution parameter $\Delta x^{QS} = \Delta x^{QS}_{Q}$. Consequently, it is necessary to decide whether the transient

![Fig. 2. 1-D problem geometry.](image-url)

<table>
<thead>
<tr>
<th>TABLE II</th>
<th>MATERIAL PARAMETERS FOR THE MEDIUM I AND MEDIUM II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>$\varepsilon_{0}$ F/m</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\mu_{0}$ H/m</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$10^{-3}$ S/m</td>
</tr>
<tr>
<td>$\tau_{r}$</td>
<td>$5.271 \times 10^{-5}$ s</td>
</tr>
<tr>
<td>$\tau_{f}$</td>
<td>$0.1342$ s</td>
</tr>
<tr>
<td>$\tau_{m}$</td>
<td>$1.8 \times 10^{10}$</td>
</tr>
<tr>
<td>$L_{x}$</td>
<td>$15811.5$ m</td>
</tr>
</tbody>
</table>
Physics of the MQS phenomenon will be degraded when the improper $\alpha$ values are used. To better illustrate this, the proper and the improper cases are considered. The time domain signal $H_z(x_{\text{fixed}}, t)$ and the spatial domain field distribution $H_z(x, t_{\text{fixed}})$ are calculated in the second layer. The rescaling factor $|\alpha'|$ is $\sim 1$ for proper $\alpha$. The proper and the improper QS-FDTD results are compared with those of analytical [13] and computationally intensive classical FDTD results.

### A. Proper Value of $\alpha$

In order to find the proper value of $\alpha$, we must consider the following condition as:

$$\alpha \ll \min(\alpha_k, \alpha_N, \alpha_{MQS}, \alpha_T)$$

corresponding to $\alpha = 9 \times 10^3$ and $\lambda_{MQS} = 1.0541 \times 10^5$. It leads to the predefined geometric resolution as $\Delta x_{MQS} = \Delta x = 3 \text{ m}$. This means that $\Delta t$ is relaxed $(9 \times 10^3)^{1/2} \cong 94.87$ times in which the MQS condition is
strongly satisfied. The numerical results shown in Fig. 3(a)–(c) are clearly revealed that chosen $\alpha$ is true. Excellent agreements are observed between the analytical, the classical FDTD, and the QS-FDTD results.

B. Improper Values of $\alpha$

In this case, two improper values are selected as equal to the minimum value of the proper $\alpha$ and ten times bigger than the minimum value of the proper $\alpha$. The results in Fig. 4(a)–(c) show how the wrong values adversely affect the numerical solution.

It is clear that the amplitude error and the phase error are not negligible for the improper values. Furthermore, they worsen for larger $\alpha$.

In Table IV, the memory and computation time requirements are compared between the methods. The memory requirements are the same due to $\Delta x = \Delta x^\text{QS}$. Superiority of the QS-FDTD method is clear for computational time.

<table>
<thead>
<tr>
<th>TABLE IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>COMPARISONS OF COMPUTATIONAL REQUIREMENTS</td>
</tr>
<tr>
<td>QS-FDTD</td>
</tr>
<tr>
<td>Memory</td>
</tr>
<tr>
<td>Computation Time</td>
</tr>
</tbody>
</table>

V. CONCLUSION AND DISCUSSION

In this paper, the MQS approximation is considered for the FDTD solution of the low-frequency electromagnetic problems. First, slowing down concept for the propagation velocity is clearly revealed with an analytical view of the MQS approximation. Fundamentals of the $\varepsilon$-scaling technique are evaluated with its effects over the physical and numerical parameters. Important and crucial points of the $\varepsilon$ (material) scaling technique in the QS-FDTD solution are clarified. Based on the analytical and numerical analysis, a set of criterion is extracted to find a proper value of the scaling parameter. In order to preserve the physics of the low-frequency MQS phenomenon, importance of the proper value of the scaling parameter is shown by the numerical results. Other parameter scaling techniques and implicit methods are also discussed. In the future, investigations are planned for the application of EQS condition, effect of exponential differencing, and anisotropy of numerical dispersion in the FDTD method.

REFERENCES


Mehmet Burak Özakın was born in Erzurum, Turkey, in 1987. He received the B.S. and M.S. degrees from the Electronics Engineering Department, Gebze Institute of Technology, Gebze, Turkey, in 2009 and 2011, respectively, where he is currently pursuing the Ph.D. degree. He is a Research Assistant with Gebze Technical University. His scientific interests focus on numerical and analytical solutions of electromagnetic problems in the time domain.

Serkan Aksoy was born in Bolu, Turkey, in 1974. He received the B.S. degree in electronics and communication engineering from Istanbul Technical University, Istanbul, Turkey, in 1996, and the M.S. and Ph.D. degrees in electronics engineering from the Gebze Institute of Technology, Gebze, Turkey, in 1999 and 2003, respectively.

He is currently with the Electronics Engineering Department, Gebze Technical University, Gebze, and a part-time Researcher involved in various projects of The Scientific and Technical Research Council of Turkey. His scientific interest is about analytical and numerical time-domain solutions of electromagnetic and acoustic problems.