Center of Gravity, Centroid, and Moment of Inertia

Chapter Objectives

✓ To discuss the concept of the center of gravity, center of mass, and the centroid.
✓ To show how to determine the location of the center of gravity and centroid for a system of discrete particles and a body of arbitrary shape.
✓ To present a method for finding the resultant of a general distributed loading.
✓ To show how to determine the moment of inertia of an area.
Center of Gravity

- Locates the resultant weight of a system of particles
- Consider system of n particles fixed within a region of space
- The weights of the particles can be replaced by a single (equivalent) resultant weight having defined point G of application
Center of Gravity

- Resultant weight = total weight of n particles
  \[ W_R = \sum W \]

- Sum of moments of weights of all the particles about x, y, z axes = moment of resultant weight about these axes

- Summing moments about the x axis,
  \[ \bar{x}W_R = \bar{x}_1W_1 + \bar{x}_2W_2 + \ldots + \bar{x}_nW_n \]

- Summing moments about the y axis,
  \[ \bar{y}W_R = \bar{y}_1W_1 + \bar{y}_2W_2 + \ldots + \bar{y}_nW_n \]
CENTER OF GRAVITY, CENTER OF MASS, AND THE CENTROID OF A BODY (cont)

Center of Gravity

• Although the weights do not produce a moment about z axis, by rotating the coordinate system 90° about x or y axis with the particles fixed in it and summing moments about the x axis,
\[
\sum \bar{z} W_R = \sum \bar{z}_1 W_1 + \sum \bar{z}_2 W_2 + \ldots + \sum \bar{z}_n W_n
\]

• Generally,
\[
\bar{x} = \frac{\sum \bar{x} m}{\sum m}; \quad \bar{y} = \frac{\sum \bar{y} m}{\sum m}; \quad \bar{z} = \frac{\sum \bar{z} m}{\sum m}
\]
Center of Mass

- Provided acceleration due to gravity $g$ for every particle is constant, then $W = mg$

$$\bar{x} = \frac{\sum \tilde{x}m}{\sum m} ; \bar{y} = \frac{\sum \tilde{y}m}{\sum m} , \bar{z} = \frac{\sum \tilde{z}m}{\sum m}$$

- By comparison, the location of the center of gravity coincides with that of center of mass

- Particles have weight only when under the influence of gravitational attraction, whereas center of mass is independent of gravity
Centroid of Mass

- A rigid body is composed of an infinite number of particles
- Consider arbitrary particle having a weight of \(dW\)

\[
\bar{x} = \frac{\int \tilde{x} dW}{\int dW}; \quad \bar{y} = \frac{\int \tilde{y} dW}{\int dW}; \quad \bar{z} = \frac{\int \tilde{z} dW}{\int dW}
\]
Centroid of a Volume

- Consider an object subdivided into volume elements $dV$, for location of the centroid,

\[
\bar{x} = \frac{\int x dV}{V}; \quad \bar{y} = \frac{\int y dV}{V}; \quad \bar{z} = \frac{\int z dV}{V}
\]
Centroid of an Area

- For centroid of surface area of an object, such as plate and shell, subdivide the area into differential elements $dA$

$$
\bar{x} = \frac{\int \tilde{x} dA}{A}; \quad \bar{y} = \frac{\int \tilde{y} dA}{A}; \quad \bar{z} = \frac{\int \tilde{z} dA}{A}
$$
Centroids of common shapes of areas

<table>
<thead>
<tr>
<th>Shape</th>
<th>$\bar{x}$</th>
<th>$\bar{y}$</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular area</td>
<td></td>
<td>$\frac{h}{3}$</td>
<td>$\frac{bh}{2}$</td>
</tr>
<tr>
<td>Quarter-circular area</td>
<td>$\frac{4r}{3\pi}$</td>
<td>$\frac{4r}{3\pi}$</td>
<td>$\frac{\pi r^2}{4}$</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>$\frac{4r}{3\pi}$</td>
<td>$\frac{\pi r^2}{2}$</td>
</tr>
<tr>
<td>Quarter-elliptical area</td>
<td>$\frac{4a}{3\pi}$</td>
<td>$\frac{4b}{3\pi}$</td>
<td>$\frac{ab}{2}$</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>$\frac{4b}{3\pi}$</td>
<td>$\frac{ab}{2}$</td>
</tr>
<tr>
<td>Semiparabolic area</td>
<td>$\frac{3a}{8}$</td>
<td>$\frac{3h}{5}$</td>
<td>$\frac{2ah}{3}$</td>
</tr>
<tr>
<td>Parabolic area</td>
<td>0</td>
<td>$\frac{3h}{5}$</td>
<td>$\frac{4ah}{3}$</td>
</tr>
<tr>
<td>Parabolic spandrel</td>
<td>$\frac{3a}{4}$</td>
<td>$\frac{3h}{10}$</td>
<td>$\frac{ah}{3}$</td>
</tr>
</tbody>
</table>
COMPOSITE BODIES

• Consists of a series of connected “simpler” shaped bodies, which may be rectangular, triangular or semicircular

• A body can be sectioned or divided into its composite parts

• Accounting for finite number of weights

\[
\bar{x} = \frac{\sum \tilde{x} W}{\sum W} \quad \bar{y} = \frac{\sum \tilde{y} W}{\sum W} \quad \bar{z} = \frac{\sum \tilde{z} W}{\sum W}
\]
COMPOSITE BODIES (cont)

Procedure for Analysis

Composite Parts
• Divide the body or object into a finite number of composite parts that have simpler shapes
• Treat the hole in composite as an additional composite part having negative weight or size

Moment Arms
• Establish the coordinate axes and determine the coordinates of the center of gravity or centroid of each part
COMPOSITE BODIES (cont)

Procedure for Analysis

Summations

• Determine the coordinates of the center of gravity by applying the center of gravity equations

• If an object is symmetrical about an axis, the centroid of the objects lies on the axis
EXAMPLE 1

Locate the centroid of the plate area.
EXAMPLE 1 (cont)

Solution

Composite Parts

• Plate divided into 3 segments.
• Area of small rectangle considered “negative”.

[Diagram of composite parts]
EXAMPLE 1 (cont)

Moment Arm

• Location of the centroid for each piece is determined and indicated in the diagram.

<table>
<thead>
<tr>
<th>Segment</th>
<th>( A (m^2) )</th>
<th>( \bar{x} ) (m)</th>
<th>( \bar{y} ) (m)</th>
<th>( \bar{x}A (m^3) )</th>
<th>( \bar{y}A (m^3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2}(3)(3) = 4.5 )</td>
<td>1</td>
<td>1</td>
<td>4.5</td>
<td>4.5</td>
</tr>
<tr>
<td>2</td>
<td>( (3)(3) = 9 )</td>
<td>-1.5</td>
<td>1.5</td>
<td>-13.5</td>
<td>13.5</td>
</tr>
<tr>
<td>3</td>
<td>( -(2)(1) = -2 )</td>
<td>-2.5</td>
<td>2</td>
<td>5</td>
<td>-4</td>
</tr>
<tr>
<td></td>
<td>( \Sigma A = 11.5 )</td>
<td></td>
<td></td>
<td>( \Sigma \bar{x}A = -4 )</td>
<td>( \Sigma \bar{y}A = 14 )</td>
</tr>
</tbody>
</table>

Summations

\[
\bar{x} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{mm}
\]

\[
\bar{y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{mm}
\]
Pressure Distribution over a Surface

- Consider the flat plate subjected to the loading function \( \rho = \rho(x, y) \) Pa (Force per unit area)
- Determine the force \( dF \) acting on the differential area \( dA \) m\(^2\) of the plate, located at the differential point \( (x, y) \)
  \[
  dF = [\rho(x, y) \text{ N/m}^2](dA \text{ m}^2)
  = [\rho(x, y) \text{ dA}] \text{ N}
  
- Entire loading represented as infinite parallel forces acting on separate differential area \( dA \)
RESULTANT OF A DISTRIBUTED LOADING (cont)

Pressure Distribution over a Surface
• This system will be simplified to a single resultant force $F_R$ acting through a unique point on the plate
RESULTANT OF A DISTRIBUTED LOADING (cont)

Magnitude of Resultant Force

- To determine magnitude of $F_R$, sum up the differential forces $dF$ acting over the plate’s entire surface area $dA$
- Magnitude of resultant force = total volume under the distributed loading diagram
- Location of Resultant Force is

$$
\bar{x} = \frac{\int_A x \rho(x, y) dA}{\int_A \rho(x, y) dA} = \frac{\int_V x dV}{\int_V dV}
$$

$$
\bar{y} = \frac{\int_A y \rho(x, y) dA}{\int_A \rho(x, y) dA} = \frac{\int_V y dV}{\int_V dV}
$$
Distributed loads on beams

\( W \) is the force per unit length, N/m

\[
W = \int_0^L w\,dx \quad W = \int_0^L dA = A
\]

Thus, the load \( W \) is equal to the area below the distributed load curve, \( W \).

The point of application of the equivalent load is the centroid of the area of the distributed load.

\[
\bar{x} = \frac{\int \tilde{x}dA}{\int dA}
\]
Distributed loads on beams

1) Find the reaction forces at the hinge A and the tension on the cable BC

\[ F = \frac{6 \cdot 3}{2} = 9 \text{kN} \]

\[ \sum M_A = 0 \]
\[ 9 \cdot 4 - T \cdot \frac{3}{6.7} \cdot 6 = 0 \]
\[ T = 13.4 \text{kN} \]

\[ \sum F_x = 0 \]
\[ P_x - 13.4 \cdot \frac{6}{6.7} = 0 \]
\[ P_x = 12 \text{kN} \]

\[ \sum F_y = 0 \]
\[ P_y - 9 + 13.4 \cdot \frac{3}{6.7} = 0 \]
\[ P_y = 3 \text{kN} \]
Example:

Wind load

\[
q = 50\sqrt{x}
\]

If the height of the building is 100m and if foundation is assumed to be fixed support, find the reaction forces exerted by the foundation.
Solution:

\[ Q = \int q(x) \, dx = \int_{0}^{100} 50\sqrt{x} \, dx = 50 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \bigg|_{0}^{100} = \frac{100}{3} \cdot 100^{\frac{3}{2}} = 333.33 N \]

\[ d = \frac{\int x \cdot q(x) \, dx}{\int q(x) \, dx} = \frac{\int_{0}^{100} x \cdot 50\sqrt{x} \, dx}{\int_{0}^{100} 50\sqrt{x} \, dx} = \frac{50 \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}}}{333.33} = \frac{50 \cdot 100^{\frac{5}{2}}}{333.33} = 60.7 \text{ m} \]
Solution (continue):

\[ \sum m_A = 0 \]

\[ M_A - 3300 \times 60.7 = 0 \]

\[ M_A = 2 \times 10^6 \text{N} \]

\[ \sum F_y = 0 \quad A_y - 3300 = 0 \]

\[ A_y = 3300 \text{N} \]

\[ \sum F_x = 0 \quad A_x = 0 \quad \text{(neglecting weight of the building!!!)} \]
Determine (a) the distributed load $w_o$ at the end D of the beam ABCD for which the reaction at B is zero, (b) the corresponding reactions at C.

Distributed loads on beams

50 kN·m

3.5 kN/m

$w_o = ?$
MOMENTS OF INERTIA FOR AREAS

• Centroid for an area is determined by the first moment of an area about an axis
• Second moment of an area is referred as the moment of inertia
• Moment of inertia of an area originates whenever one relates the normal stress $\sigma$ or force per unit area
Moment of Inertia

- Consider area A lying in the x-y plane
- By definition, moments of inertia of the differential plane area dA about the x and y axes

\[ dI_x = y^2 dA \quad dI_y = x^2 dA \]

- For entire area, moments of inertia are given by

\[ I_x = \int_A y^2 dA \]
\[ I_y = \int_A x^2 dA \]
Moment of Inertia

- Formulate the second moment of $dA$ about the pole $O$ or $z$ axis
- This is known as the polar axis
  \[ dJ_O = r^2 dA \]
  where $r$ is perpendicular from the pole ($z$ axis) to the element $dA$
- Polar moment of inertia for entire area,
  \[ J_O = \int_A r^2 dA = I_x + I_y \]
PARALLEL AXIS THEOREM FOR AN AREA

- For moment of inertia of an area known about an axis passing through its centroid, determine the moment of inertia of area about a corresponding parallel axis using the parallel axis theorem.
- Consider moment of inertia of the shaded area.
- A differential element $dA$ is located at an arbitrary distance $y'$ from the centroidal $x'$ axis.
PARALLEL AXIS THEOREM FOR AN AREA (cont)

- The fixed distance between the parallel x and x' axes is defined as $d_y$
- For moment of inertia of $dA$ about x axis
  $$dI_x = \left(y' + d_y\right)^2 dA$$
- For entire area
  $$I_x = \int_A \left(y' + d_y\right)^2 dA$$
  $$= \int_A y'^2 dA + 2d_y \int_A y' dA + d_y^2 \int_A dA$$
- First integral represent the moment of inertia of the area about the centroidal axis
PARALLEL AXIS THEOREM FOR AN AREA (cont)

- Second integral $= 0$ since $x'$ passes through the area’s centroid $C$
  $$\int y' \, dA = \bar{y} \int dA = 0; \quad \bar{y} = 0$$

- Third integral represents the total area $A$
  $$I_x = \bar{I}_x + Ad_y^2$$

- Similarly
  $$I_y = \bar{I}_y + Ad_x^2$$

- For polar moment of inertia about an axis perpendicular to the $x$-$y$ plane and passing through pole $O$ (z axis)
  $$J_o = \bar{J}_c + Ad_z^2$$
EXAMPLE 2

Determine the moment of inertia for the rectangular area with respect to (a) the centroidal $x'$ axis, (b) the axis $x_b$ passing through the base of the rectangular, and (c) the pole or $z'$ axis perpendicular to the $x'$-$y'$ plane and passing through the centroid $C$. 

![Diagram of a rectangular area with axes labeled $x'$, $y'$, $x_b$, and centroid C.]
EXAMPLE 2 (cont)

Solution

Part (a)
• Differential element chosen, distance \( y' \) from \( x' \) axis.
• Since \( dA = b \ dy' \),

\[
\bar{I}_x = \int_A y'^2 \, dA = \int_{-h/2}^{h/2} y'^2 \, (bdy') = \int_{-h/2}^{h/2} y'^2 \, dy = \frac{1}{12} bh^3
\]

Part (b)
• By applying parallel axis theorem,

\[
I_{xb} = \bar{I}_x + Ad^2 = \frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3} bh^3
\]
EXAMPLE 2 (cont)

Solution

Part (c)

- For polar moment of inertia about point C,

\[
\bar{I}_y = \frac{1}{12} hb^3
\]

\[
J_C = \bar{I}_x + \bar{I}_y = \frac{1}{12} bh(h^2 + b^2)
\]
MOMENTS OF INERTIA FOR COMPOSITE AREAS

- Composite area consist of a series of connected simpler parts or shapes
- Moment of inertia of the composite area = algebraic sum of the moments of inertia of all its parts

Procedure for Analysis

Composite Parts
- Divide area into its composite parts and indicate the centroid of each part to the reference axis

Parallel Axis Theorem
- Moment of inertia of each part is determined about its centroidal axis
MOMENTS OF INERTIA FOR COMPOSITE AREAS (cont)

Procedure for Analysis

Parallel Axis Theorem

• When centroidal axis does not coincide with the reference axis, the parallel axis theorem is used

Summation

• Moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts
EXAMPLE 3

Compute the moment of inertia of the composite area about the x axis.
EXAMPLE 3 (cont)

Solution

Composite Parts

• Composite area obtained by subtracting the circle form the rectangle.

• Centroid of each area is located in the figure below.
EXAMPLE 3 (cont)

Solution

Parallel Axis Theorem

Circle

\[ I_x = \bar{I}_x + Ad_y^2 \]
\[ = \frac{1}{4} \pi (25)^4 + \pi (25)^2 (75)^2 = 11.4 \times 10^6 \, mm^4 \]

Rectangle

\[ I_x = \bar{I}_x + Ad_y^2 \]
\[ = \frac{1}{12} (100)(150)^3 + (100)(150)(75)^2 = 112.5 \times 10^6 \, mm^4 \]
EXAMPLE 3 (cont)

Solution

Summation

For moment of inertia for the composite area,

\[ I_x = -11.4 \times 10^6 + 112.5 \times 10^6 \]

\[ = 101 \times 10^6 \text{mm}^4 \]