Design Example:

For the system with the following block diagram representation

\[ R(s) \rightarrow + \rightarrow G_c(s) \rightarrow \frac{1}{s(s+1)} \rightarrow Y(s) \]

Find \( G_c(s) \) so that the dominant closed loop poles are at \(-3 \pm j3\)
Solution: Start with \( G_c(s) = K \) \( \Rightarrow \) \( \Delta_{cl} = 1 + K \frac{1}{s(s + 1)} \)

Compensator design with \( G_c(s) = K \) is not sufficient
However if the compensator is in the form

\[ G_c(s) = K \frac{s + a}{s + b} \quad \text{with} \quad b > a > 0 \]

\{ \text{Lead Compensation} \}

The closed loop Char. Eq. will be in the form

\[ \Delta_{cl} = 1 + K \frac{s + a}{s + b} \frac{1}{s(s + 1)} \]

Centriod at

\[ \frac{-b - 0 - 1 + a}{2} \]

3 closed loop poles
(an additional at \( c \))
By changing the values of \( a \) and \( b \) we can change the place of the centroid, and hence bend the root locus to go through the desired closed loop poles.

Note that we need to ensure \( -3 \pm j3 \) are more dominant compared to the new pole located at \( c \).

The problem is now to design \( K, a \) and \( b \) to ensure the closed loop dominant poles are at \( -3 \pm j3 \).
Method 1:

- Using Magnitude and Angle conditions

The closed loop transfer function:

\[ H(s) = \frac{K(s+a)}{s(s+b)(s+1)} \left(1 + \frac{K(s+a)}{s(s+b)(s+1)}\right) \]

Characteristic equation:

\[ \Delta = 1 + \frac{K(s + a)}{s(s + b)(s + 1)} \]

Magnitude condition dictates

\[ \Delta|_{s=-3+j3} = 1 + K \frac{s + a}{s(s + b)(s + 1)} \bigg|_{s=-3+j3} = 0 \]
Also from Angle condition

$$\left. \frac{s + a}{s (s + b) (s + 1)} \right|_{s=-3+j3} = -180$$

select $a = 3$ in order to reduce the unknowns, now we have

$$\left. - \text{Angle} (s + b) \right|_{s=-3+j3} = -180 + \left. \text{Angle} \left( \frac{s (s + 1)}{(s + 3)} \right) \right|_{s=-3+j3}$$

or simply

$$\left. - \tan^{-1} \left( \frac{3}{b - 3} \right) \right| = -11.3^\circ \quad b = 18$$
How did we selected 'a'?

**Selection of 'a'**

- There is no unique value for 'a'
- If it is selected to big, we might not be able to find an appropriate value for 'b'
- If it is picked to small, the extra pole inserted might become the dominant pole
• Back to the Magnitude condition with the values of \( a \) and \( b \) selected

\[
\Delta|_{s= -3 + j3} = 1 + K \frac{s + 3}{s(s + 18)(s + 1)} \bigg|_{s= -3 + j3} = 0
\]

• Now we can calculate the value for \( K \)

\[
K \frac{|j3|}{|-3 + j3||15 + j3||-2 + j3|} \bigg|_{s= -3 + j3} = 1
\]

\[
K \frac{3}{\sqrt{18}\sqrt{234}\sqrt{13}} = 1
\]

\[
K = 78
\]

The equation of the compensator

\[
G_c(s) = 78 \frac{s + 3}{s + 18}
\]
Method 2:

- Coefficient matching

From the root locus we know that there are 3 closed loop poles

\[ H(s) = \frac{K(s+a)}{s(s+b)(s+1)} = \frac{K(s+a)}{1 + \frac{K(s+a)}{s(s+b)(s+1)}} = \frac{K(s+a)}{s(s+1)(s+b) + K(s+a)} \]

the closed loop ch. equation

\[ \Delta_{cl} = s^3 + (b+1)s^2 + (b+K)s + aK \]

the desired ch. equation

\[ \Delta_d = (s+c)(s+3-j3)(s+3+j3) \]
\[ = s^3 + (6+c)s^2 + (18 + 6c)s + 18c \]
• From $\Delta_{cl} = \Delta_d$ we have

\[
\begin{align*}
6 + c &= b + 1 \\
18 + 6c &= b + K \\
aK &= 18c
\end{align*}
\]

4 unknown
3 equations

Select $a=3$

then

\[
\begin{align*}
K - 6c &= 0 \\
-b + c &= -5 \\
K + b - 6c &= 18
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & -6 \\
0 & -1 & 1 \\
1 & 1 & -6
\end{bmatrix}
\begin{bmatrix}
K \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
-5 \\
18
\end{bmatrix}
\]

Solution is

\[
\begin{bmatrix}
K \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
78 \\
18 \\
13
\end{bmatrix}
\]

Same result with the value of $c$ also obtained
A side note

- From the previous solution we can determine how big the value of 'a' can be selected.

We had

\[ 5 + c = b \quad b = 6c + 18 - K \quad K = \left(\frac{18}{a}\right)c \]

\[ 5 + c = 6c + 18 - \left(\frac{18}{a}\right)c \]

when simplified

\[ c\left(\frac{18}{a} - 5\right) = 13 \]

\[ \left\{ \begin{array}{l}
\text{for } c \text{ to be positive} \\
 a < \frac{18}{5}
\end{array} \right. \]
Final value \[ \lim_{s \to 0} sY(s) = \frac{78(3)}{(13)(18)} = 1 \]

Initial value \[ \lim_{s \to \infty} sY(s) = K_1 = 1 \]
The time domain behavior

How does the compensated system behaves?

The compensated overall transfer function

\[ H(s) = \frac{78(s + 3)}{(s + 13)((s + 3)^2 + 3^2)} \]

for a step input we would have

\[ y(t) = K_1 + K_2 e^{-13t} + K_3 e^{-3t} \cos(3t) + K_4 e^{-3t} \sin(3t) \]

use PFE to find the values of \( K_i \ldots i = 1, 2, 3, 4 \)
How to decrease the steady state error (SSE) so that the output goes close to the input?

Consider

\[ R(s) + \frac{1}{s} \rightarrow K \rightarrow \frac{1}{(s + 5)(s + 1)} \rightarrow Y(s) \]

Let

\[ R(s) = \frac{1}{s} \Rightarrow \frac{Y(s)}{R(s)} = \frac{K \frac{1}{(s+1)(s+5)}}{1 + K \frac{1}{(s+1)(s+5)}} \]

Transfer function

error

\[ E(s) = R(s) - Y(s) = R(s) - \frac{K \frac{1}{(s+1)(s+5)}}{1 + K \frac{1}{(s+1)(s+5)}} R(s) \]
For a step input the SSE becomes

\[
SSE = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \left( \frac{1}{1 + K \left( \frac{1}{(s+1)(s+5)} \right)} \right) R(s)
\]

\[
= \frac{1}{1 + \frac{K}{5}}
\]

SSE goes down as K increases...
Transient Responce + SSE ?

Can we place the closed poles at desired values and decrease SSE at the same time ??

Back to the same example :

Given
\[
\frac{Y(s)}{R(s)} = \frac{K \frac{1}{(s+1)(s+5)}}{1 + K \frac{1}{(s+1)(s+5)}}
\]

Find \( K \) so that the closed loop poles are at \(-3 \pm j3\)
\[ \Delta_{cl} = 1 + \frac{K}{(s + 1)(s + 5)} \]

Root locus
Magnitude condition

\[
\left. \left( K \frac{1}{(s + 1)(s + 5)} \right) \right|_{s = -3 + 3j} = 1 \quad \Rightarrow \quad K = 13
\]

SSE

\[
= \frac{1}{1 + \frac{K}{5}} = \frac{5}{18}
\]

\[
\text{Fixed for specific value of } K = 13
\]

- The value of SSE is fixed. Then how to we shape the transient response (place dominant poles) and reduce SSE at the same time??
Lag Compensation

This time; consider

\[ R(s) \quad + \quad G_c(s) \quad K \quad G_p(s) \quad Y(s) \]

where

\[ G_p(s) = \frac{1}{(s + 5)(s + 1)} \quad \text{and let} \quad K = 13 \quad \text{to meet transient specs} \]

Design

\[ G_c(s) = \frac{s + c}{s + d}, \quad c > d > 0 \quad \text{typical values for} \quad c, d \quad \text{are} \]

\[ c = 0.1 \quad \quad d = 0.01 \]

Error (from the block diagram)

\[ E(s) = \frac{1}{1 + KG_pG_c} \]
The SSE is then (with unit step input)

\[
\text{SSE} = \lim_{s \to 0} sE(s) = s \frac{1}{s} \frac{1}{1 + KG_cG_p}
\]

for our case

\[
= \lim_{s \to 0} \frac{1}{1 + 13 \left( \frac{s+0.1}{s+0.01} \right) \left( \frac{1}{(s+1)(s+5)} \right)}
\]

\[
\text{SSE} = \frac{1}{1 + \frac{130}{5}} = \frac{1}{27} \quad \text{(approximately)}
\]

- When we have \( G_c(s) = 1 \) and \( K=13 \) \( \text{SSE} = 1/3 \)
- The Lag compensator have improved the SSE performance !!!!
Example

Design $G_c(s)$ so that
- Dominant closed loop poles are at $s = -4 \pm j5$
- The SSE is 0.01 for a step input

with

$$G_p(s) = \frac{1}{(s + 5)(s + 1)}$$
Method

- Design a Lead compensator to place the dominant poles at the desired places while neglecting the effects of the Lag compensator. Then design the Lag compensator to meet the SSE specifications.

Solution:

let \( G_c(s) = \frac{K(s + a)}{(s + b)} \) \{ The closed loop TF \( \frac{G_c G_p}{1 + G_c G_p} \)

ch.eq. \( \Delta(s) = 1 + K \left( \frac{s + a}{s + b} \right) \left( \frac{1}{(s + 1)(s + 5)} \right) \)

(required for Root Locus)
The Root Locus

Need to find the values of $a$ and $b$
Angle condition

\[\text{angle} \left[ \frac{s + a}{(s + b)(s + 1)(s + 5)} \right]_{s = -4 + j5} = -180\]

pick

\[a = 10\]  \(\text{Guess why ??}\)

then

\[-\tan^{-1} \left( \frac{5}{-4 + b} \right) = -19.3 \quad \Rightarrow \quad b = 18.3\]

Magnitude Condition

\[K \left| \frac{s + 10}{(s + 18.3)(s + 1)(s + 5)} \right|_{s = -4 + j5} = 1\]

\[K = \left| \frac{(s + 18.3)(s + 1)(s + 5)}{s + 10} \right|_{s = -4 + j5} = 56.6\]
Lag Compensator Design

\[ G_c(s) = 56.6 \left( \frac{s + 10}{s + 18.3} \right) \frac{s + c}{s + d} \]

**SSE**

\[
\lim_{s \to 0} s \frac{1}{1 + G_c G_p} R(s) = 0.01
\]

That is for a step input

\[
= \frac{1}{1 + \frac{(56.6)10}{1+(18.3)(1)(5)} \frac{c}{d}} \quad \Rightarrow \quad \frac{c}{d} = 15.7
\]

**pick** \( c = 0.1 \)

\[ d = 0.00625 \]

\[ G_c(s) = 56.6 \left( \frac{s + 10}{s + 18.3} \right) \frac{s + 0.1}{s + 0.00625} \]
How accurate is our design?

With the designed compensator we have

\[ \Delta(s) = \left[ 1 + 56.6 \left( \frac{s + 10}{s + 18.3} \right) \frac{s + 0.1}{s + 0.00625} \left( \frac{1}{(s + 1)(s + 5)} \right) \right] \bigg|_{s=-4+j5} = 0 \]

Angle condition

\[ \text{angle} \left\{ \left( \frac{s + 10}{s + 18.3} \right) \frac{s + 0.1}{s + 0.00625} \left( \frac{1}{(s + 1)(s + 5)} \right) \right\} \bigg|_{s=-4+j5} = -179.798 \]

Magnitude condition

\[ \left| \left( \frac{s + 10}{s + 18.3} \right) \frac{s + 0.1}{s + 0.00625} \left( \frac{1}{(s + 1)(s + 5)} \right) \right| \bigg|_{s=-4+j5} = 0.973 \]
The reason for small offsets is

\[
\left( \frac{s + 0.1}{s + 0.00625} \right) \bigg|_{s=-4+j5} = 0.9819 \langle -0.66 \rangle
\]

Hence by selecting the lag compensator zero close to jw axis and lag compensator pole relatively close to the lag compensator zero, the overall designing changed the angle and magnitude conditions very little.

The above approximation seems fairly accurate.
Example

For the system of the form

\[ R(s) \xrightarrow{+} G_c(s) \xrightarrow{-} Y(s) \]

with

\[ G_p(s) = \frac{1}{(s + 1)(s + 3)(s + 10)} \]

Design a lead/lag compensator that has

- Dominant closed-loop poles at \( s = -2 \pm j2 \)
- \( SSE = 0.01 \) for a step input
• Start with the design of the lead compensator
  (neglect the lag compensator part for now)

\[ G_c(s) = \frac{K(s + a)}{(s + b)} \]

\[ \Delta(s) = 1 + K \left( \frac{s + a}{s + b} \right) \left( \frac{1}{(s + 1)(s + 3)(s + 10)} \right) \]

Draw the root locus
Use Angle condition to select "a"

\[
\text{angle } \left[ \frac{s + a}{(s + b) (s + 1) (s + 3) (s + 10)} \right]_{s=-2+j2} = -180
\]

Let \( a = 3.5 \quad \rightarrow \quad b = 4.38 \)

Now apply Magnitude condition to find the value of \( K \)

\[
K \left| \frac{s + 3.5}{(s + 4.38) (s + 1) (s + 3) (s + 10)} \right|_{s=-2+j2} = 1 \quad \rightarrow \quad K = 50
\]

\[
G_{\text{lead}}(s) = \frac{50(s + 3.5)}{(s + 4.38)}
\]
Now designing the lag compensator part

\[ G_c(s) = 50 \left( \frac{s + 3.5}{s + 4.38} \right) \frac{s + c}{s + d} \]

use SSE for the design of the Lag compensator parameters

\[ SSE = \lim_{s \to 0} s \frac{1}{1 + G_c G_p} R(s) = 0.01 \]

\[ 0.01 = \frac{1}{1 + \frac{1}{1 + \frac{(50)3.5}{1+(4.38)(30)}} \frac{c}{d}} \rightarrow \frac{c}{d} = 74.32 \]

Let \( c = 0.1 \) then \( d = 0.001345 \)
Overall Compensator

\[ G_c(s) = 50 \left( \frac{s + 3.5}{s + 4.38} \right) \frac{s + 0.1}{s + 0.001345} \]

check the angle and magnitude conditions to ensure the accuracy of the design

\[
\text{angle} \left\{ \left( \frac{s + 3.5}{s + 4.38} \right) \frac{s + 0.1}{s + 0.001345} \left( \frac{1}{(s + 1)(s + 3)(s + 10)} \right) \right\} \bigg|_{s = -2 + j2} = -182.398
\]

\[
50 \left| \left( \frac{s + 3.5}{s + 4.38} \right) \frac{s + 0.1}{s + 0.001345} \left( \frac{1}{(s + 1)(s + 3)(s + 10)} \right) \right| \bigg|_{s = -2 + j2} = 0.9034
\]
How does the system behave now?

Lead/Lag Compensator

Lead Compensator Only

\{ \text{output response to a step input} \}

Note the \textit{similarities} of the \textit{transient} responses.
Lead/Lag Compensator

Lead Compensator Only

output response to a step input

Note the **differences** of the **steady state** responses