Time Response

After the engineer obtains a mathematical representation of a subsystem, the subsystem is analyzed for its transient and steady-state responses to see if these characteristics yield the desired behavior.

This section is devoted to the analysis of system transient response.

This week we will concentrate on the following topics

- Finding the time response from transfer function
- Use poles and zeros to determine the time response
- Describe quantatively the transient response of first and second order systems

Poles, Zeros and System Response

In general a rational transfer function for a continous LTI system is in the form

$$H(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{s^N + \sum_{n=0}^{N-1} a_n s^n}$$

The roots of the numerator are the **zeros** of the system

$$s = \{\beta_m | m \in 1, ..., M\} \ni P(s)|_{s=\beta_m} = 0$$

and the roots of the denumerator are the **poles**

$$s = \{\alpha_n | n \in 1, ..., N\} \ni Q(s)|_{s=\alpha_n} = 0$$

System Response

From direct application of partial fraction expansion a transfer function can also be written as follows

$$\frac{\sum_{m=0}^{M} b_m s^m}{s^N + \sum_{n=0}^{N-1} a_n s^n} = \boxed{\frac{K_1}{s-\alpha_1}} + \dots + \boxed{\frac{K_p}{s^2 + 2\zeta\omega_n s + (\omega_n)^2}} + \dots$$

first order systems

second-order systems

Therefore, we can conclude that a general transfer function can be composed to first-order and second-order systems

Poles, Zeros and System Response

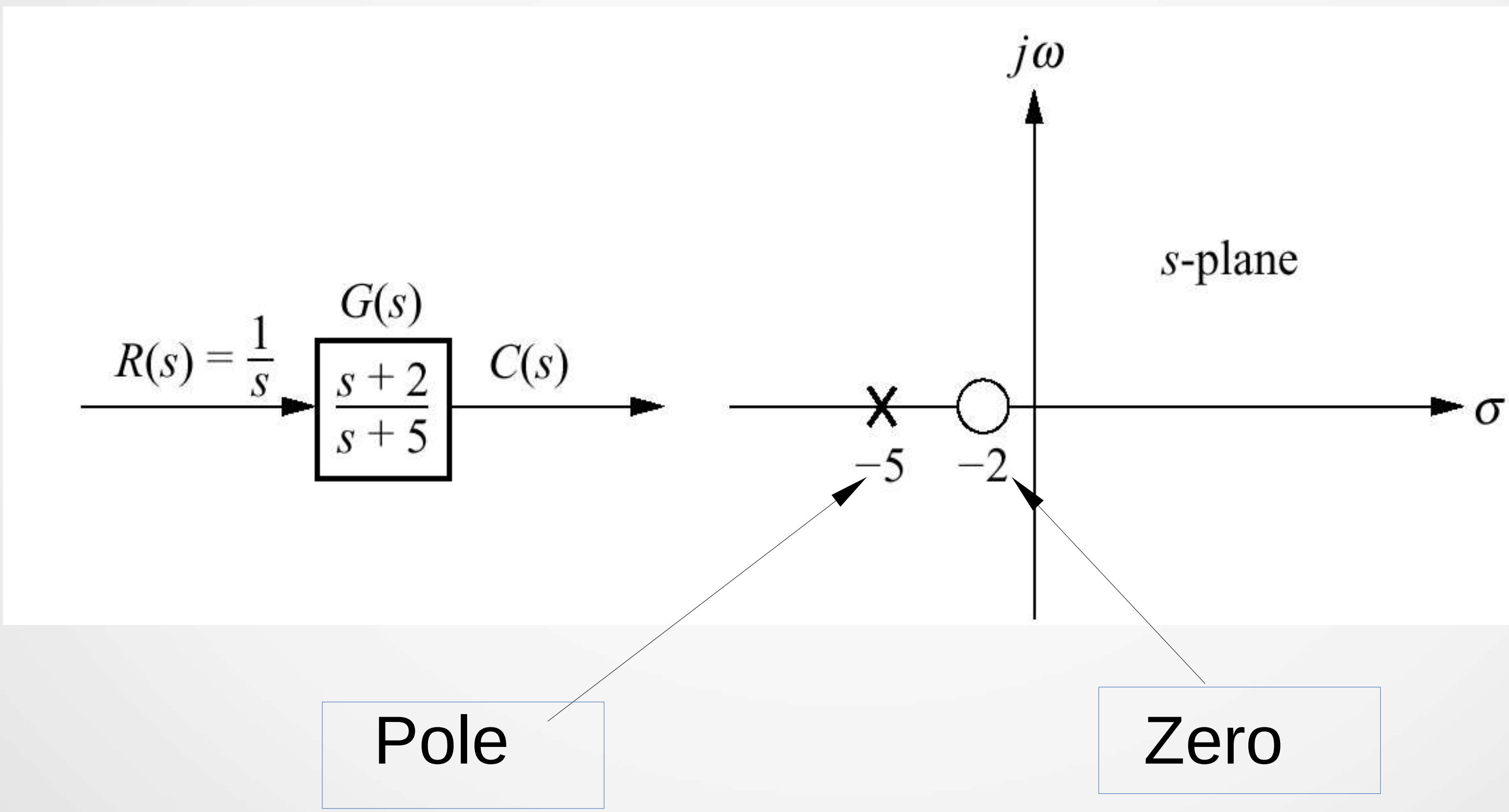
The output response of a system is the sum of two responses: the **forced response** and the **natural response**.

- The forced response is also called the steady-state response or particular solution.
- The natural response is also called the homogeneous solution.

The use of poles and zeros and their relationship to the time response of a system is a technique which allows us to simplify the evaluation of a system's response

Example (1st Order System)

Consider the following system with the given input



Example (cont.)

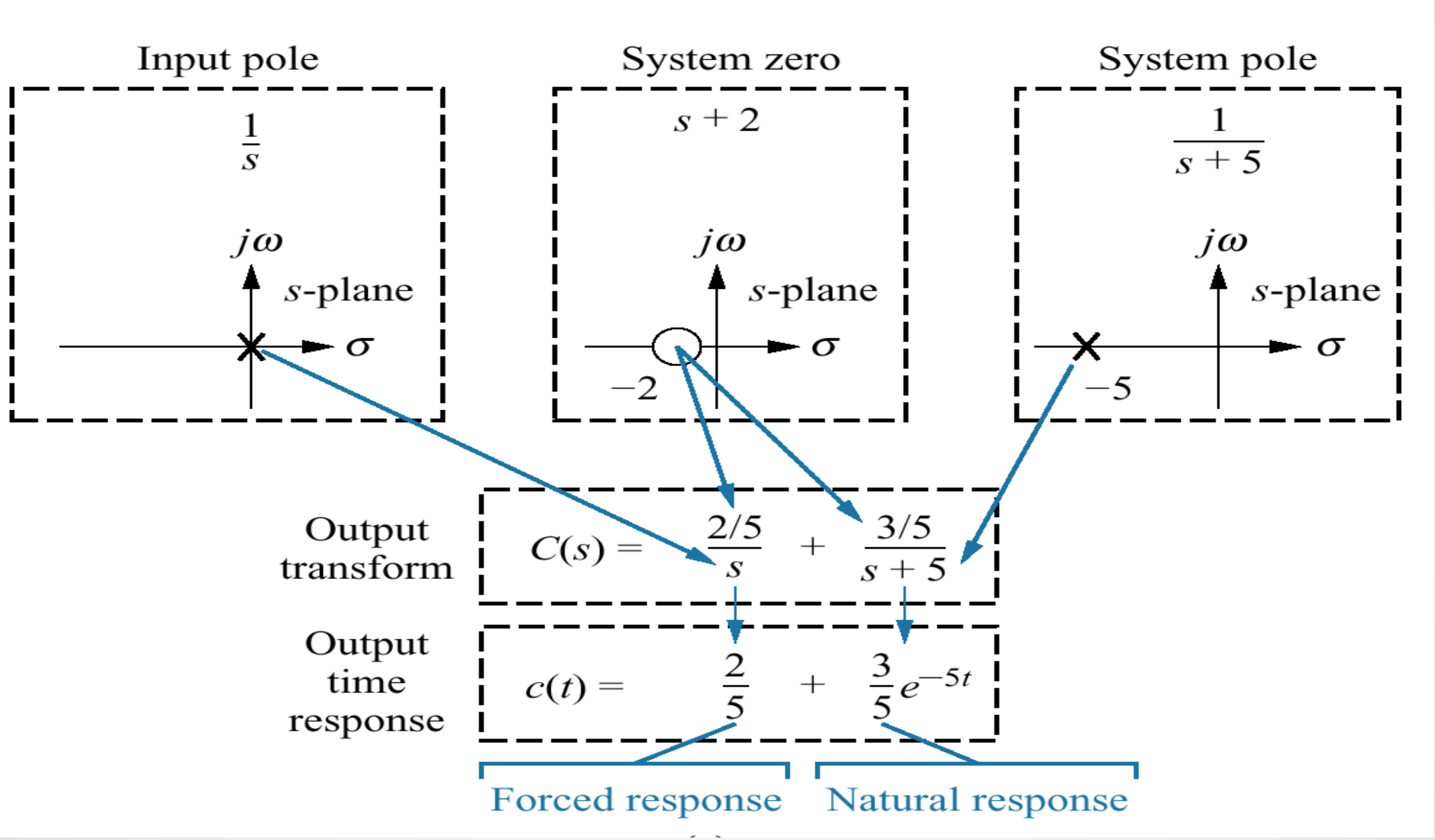
To illustrate the properties of the poles and zeros, let us find the unit step response of the system. Multiplying the transfer function by a step function (input) yields

$$C(s) = \frac{s+2}{s(s+5)} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

applying inverse Laplace transform we can obtain the time domain solution in the form

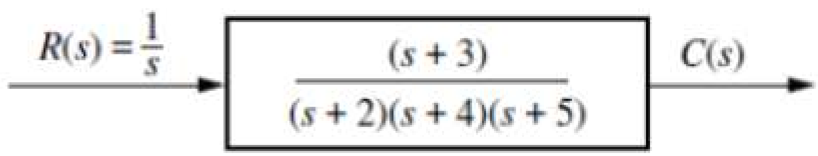
$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

Example (graphical illustration)



Example (effects of poles)

Specify the forced and natural parts of the output for the given system



Solution : By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}}$$

$$\xrightarrow{\mathcal{L}^{-1}}$$

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_{2e^{-2t}} + K_{3e^{-4t}} + K_{4e^{-5t}}}_{\text{Natural response}}$$

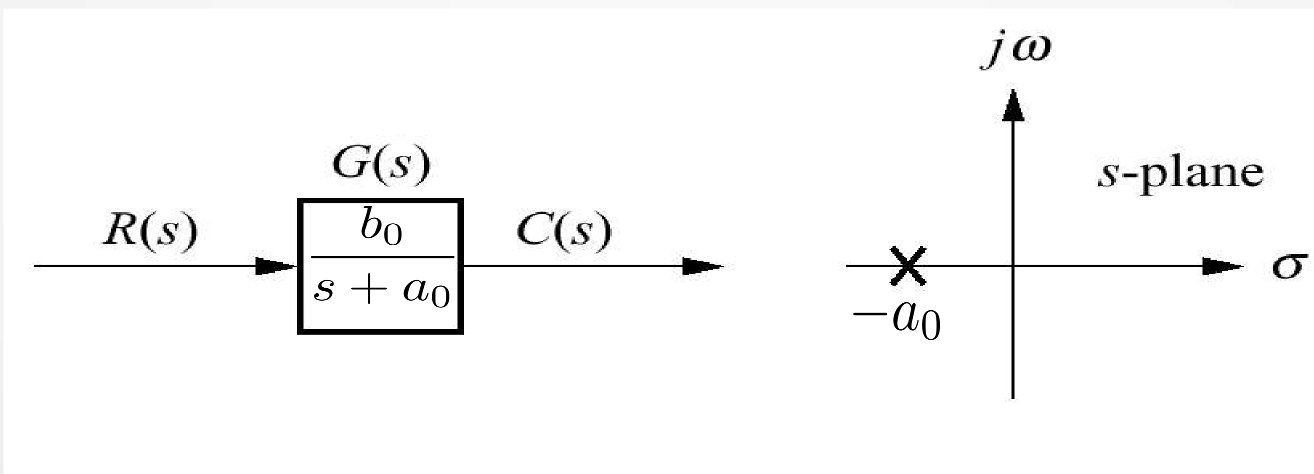
First Order Systems

A first order system is in the form

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0}{s + a_0} = \frac{K}{\tau s + 1}$$

Gain (K)

Time Constant (τ)

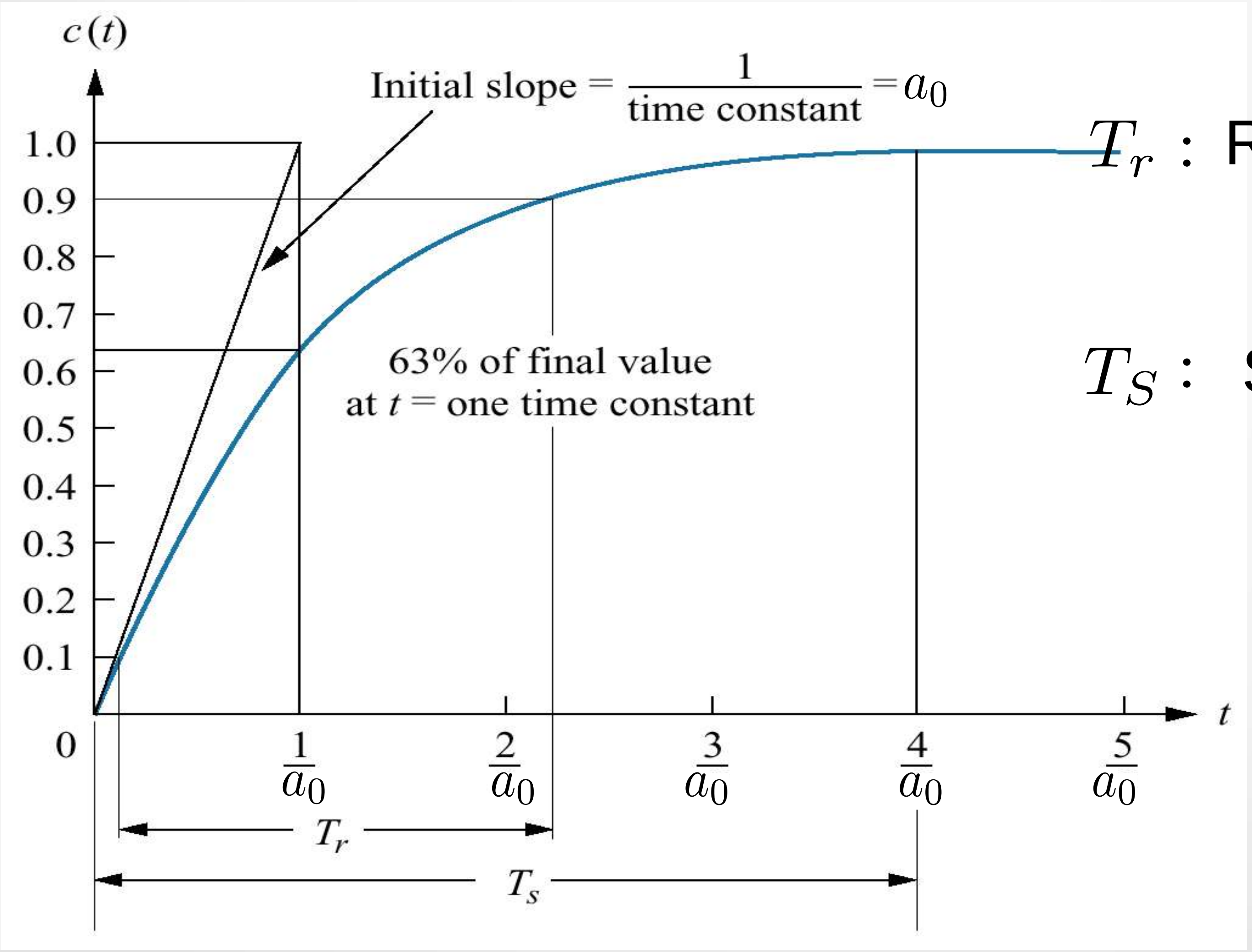


where

$$K = \frac{b_0}{a_0} \qquad \tau = \frac{1}{a_0}$$

First Order Systems

The response of a first order system to a unit step



T_r : Rise Time

T_S : Settling Time

Second Order Systems

General form of second order systems

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0} \quad \text{or} \quad G(s) = \frac{K\left(\omega_n\right)^2}{s^2 + 2\zeta\omega_n s + \left(\omega_n\right)^2}$$

where ω_n is the natural frequency and ζ is the damping ratio.

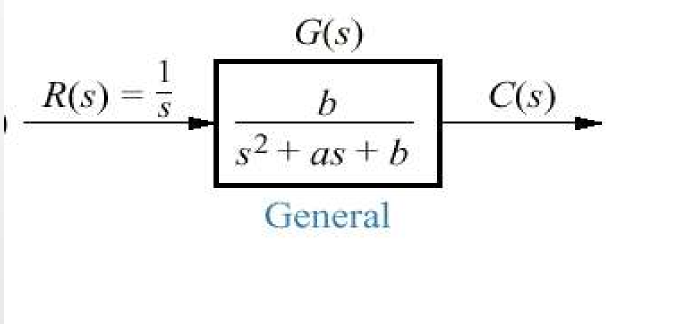
Solving for the poles of the transfer function we obtain

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

ζ determines how much cosine and sine action an output might have

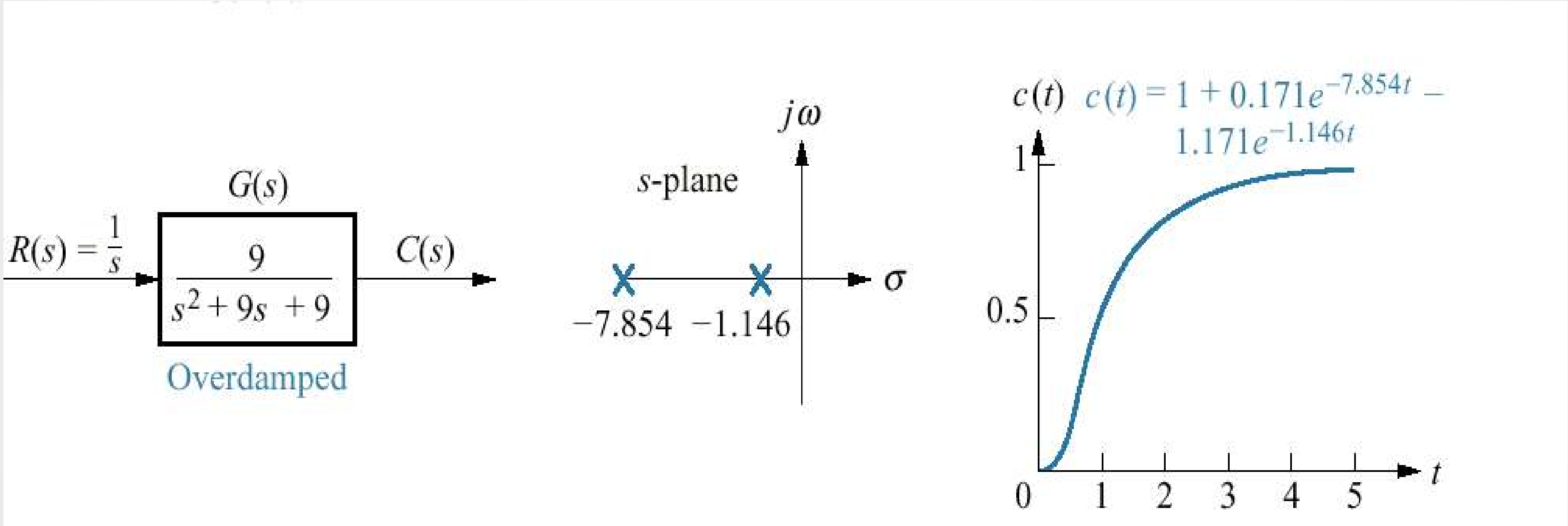
Second Order Systems

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Whereas varying a first-order system's parameter simply changes the speed of the response, changes in the parameters of a second-order system can change the form of the response.
- Before formalizing our discussion, lets take a look at numerical examples of the second-order system responses for the general sistem shown in Figure



Overdamped Response

Consider the system of the form



The output can be calculated as

$$C\left(s\right)=\frac{9}{s\left(s^{2}+9s+9\right)}=\frac{K_{1}}{s}+\frac{K_{2}}{s+7.854}+\frac{K_{3}}{s+1.146}$$

Overdamped Response (cont.)

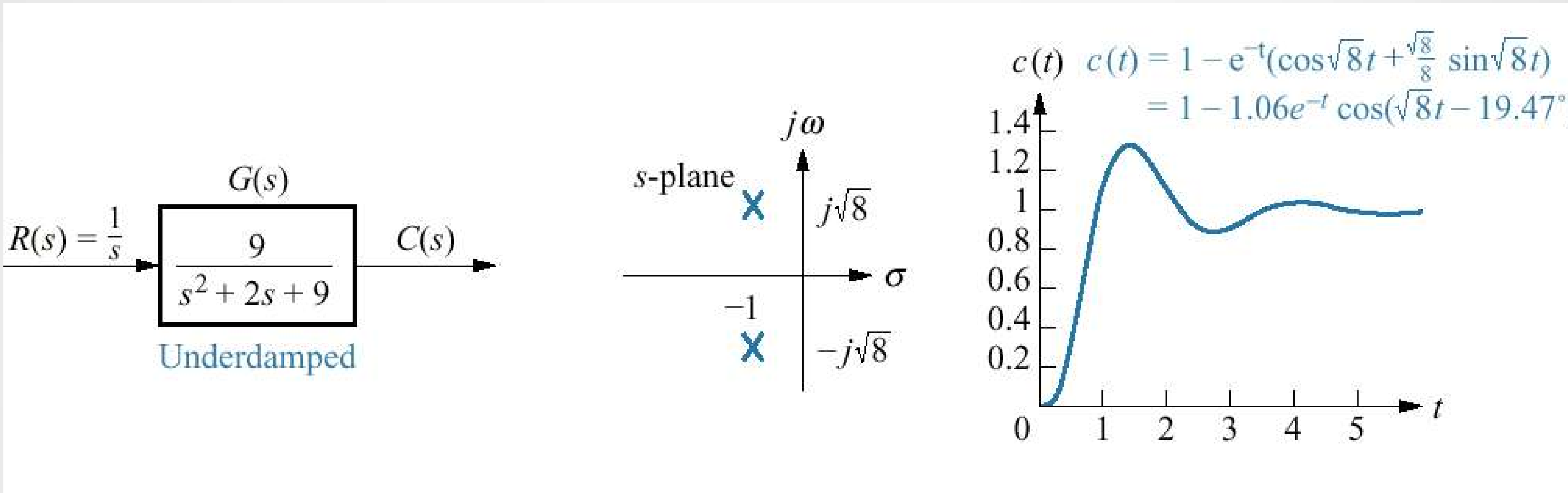
- This function has a pole at the origin that comes from the unit step input and two real poles that come from the system.
- The input pole at the origin generates the constant forced response; each of the two system poles on the real axis generates an exponential natural response whose exponential frequency is equal to the pole location.
- The output, even without calculations can be written as

$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$

- This response is called overdamped.
- Notice that the poles tell us the form of the response without the tedious calculation of the inverse Laplace transform.

Underdamped Response

This time consider

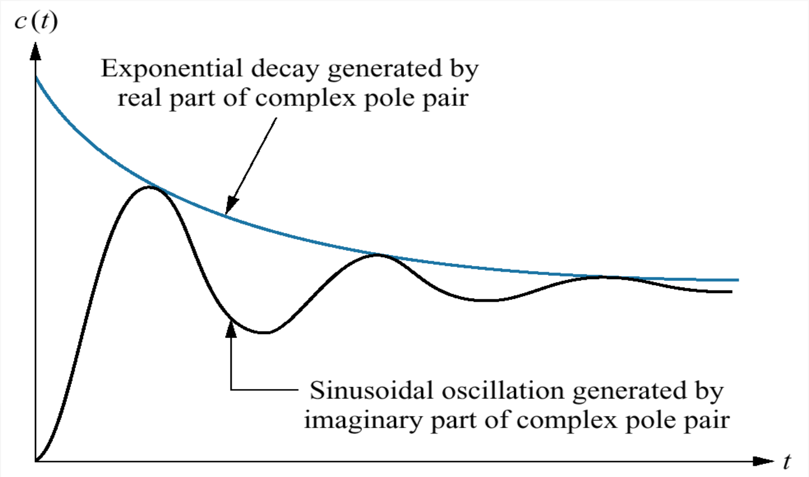


the poles are both imaginary and the time response is

$$c(t) = 1 - e^{-t(\cos(\sqrt{8}t) + \frac{\sqrt{8}}{8}\sin(\sqrt{8}t))} = 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47^o)$$

Underdamped Response (cont.)

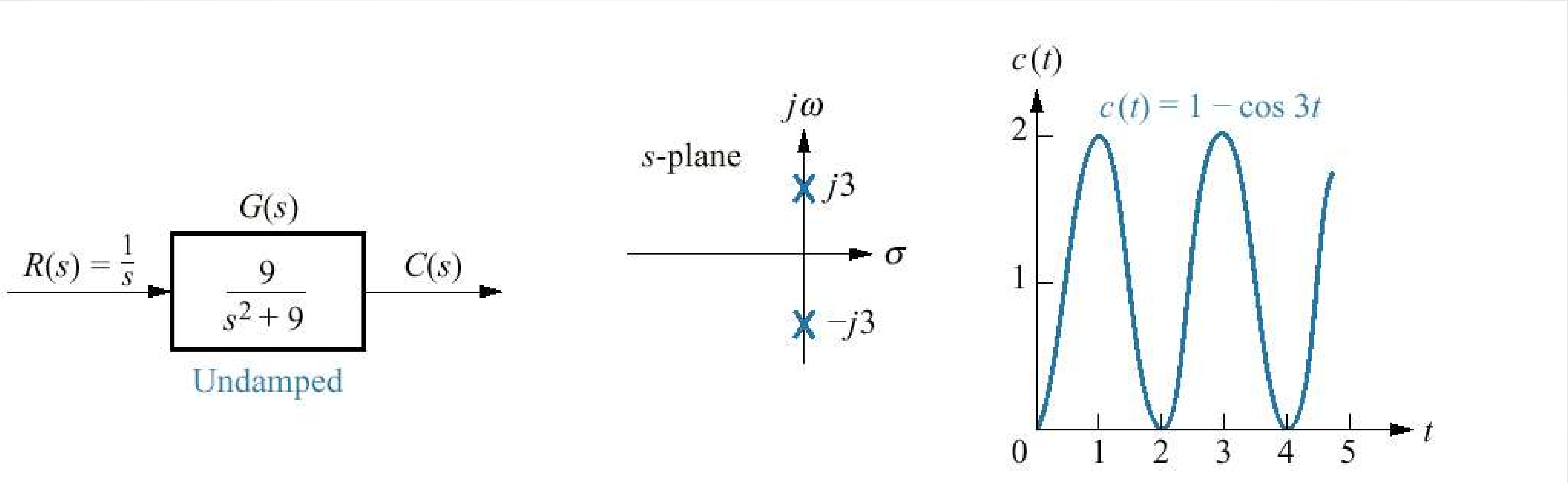
- Observe that the real part of the pole matches the exponential decay frequency of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.



- The transient response consists of an exponentially decaying amplitude generated by the real part of the system pole times a sinusoidal waveform generated by the imaginary part of the system pole.
- The time constant of the exponential decay is equal to the reciprocal of the real part of the system pole. The value of the imaginary part is the actual frequency of the sinusoid.

Undamped Response

- For this case consider



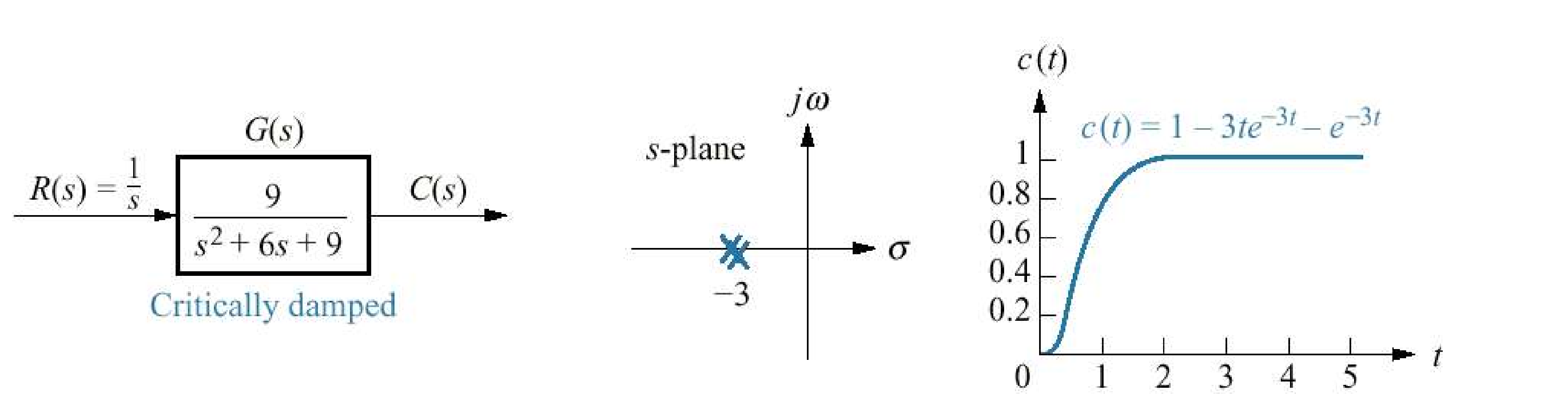
where the poles are pure imaginary and the time domain solution contains at least one undamped sinusoid.

For our example poles are at $+/-j3$ and the solution is

$$c\left(t\right)=K_1+K_2\cos\left(3t-\phi\right)$$

Critically Damped Response

- This is the case of multiple real poles to a step input



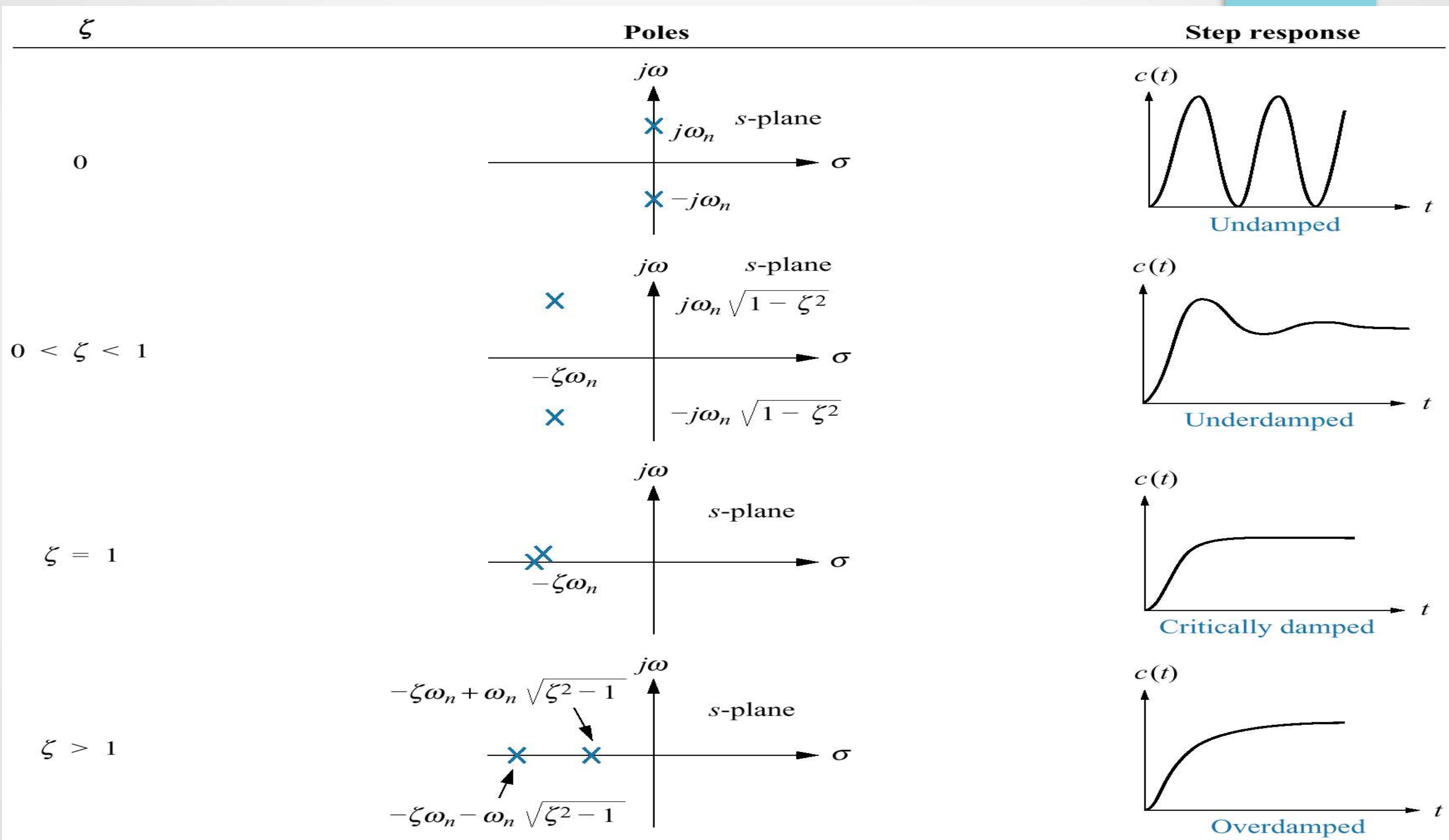
The output

$$C(s) = \frac{9}{s(s+3)^2} = \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+3)^2}$$

Hence the output can be estimated to have the form

$$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$$

Response as a function of ζ



$$G(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Step Response of 2nd Order Systems

When a step function is applied to a general 2nd order system the output will be of the form

$$C(s) = \frac{1}{s}\frac{(\omega_n)^2}{s^2 + 2\zeta\omega_n s + (\omega_n)^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + (\omega_n)^2}$$

the corresponding time domain signal is obtained after some mathematical manipulations as

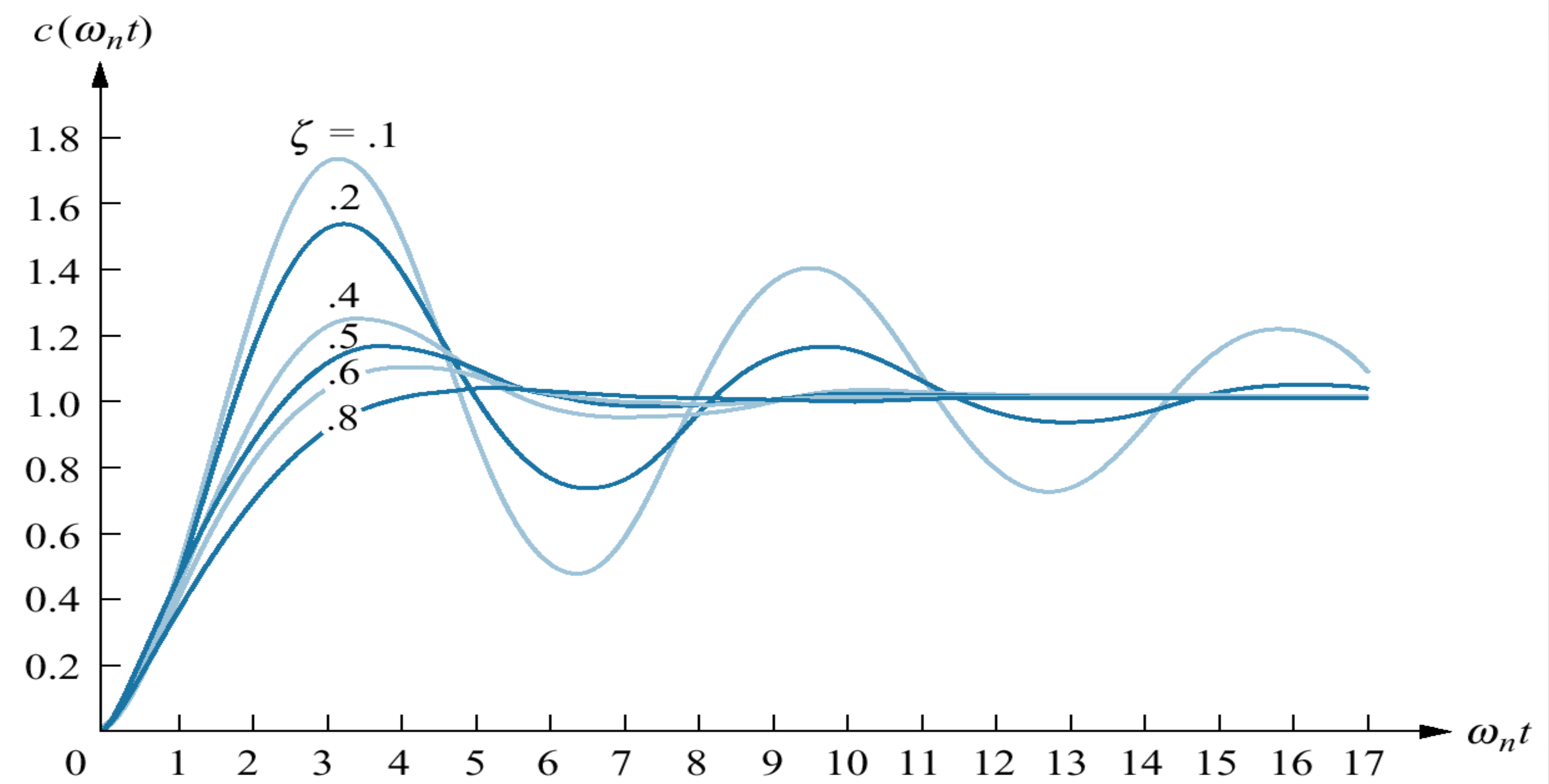
$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}\, t - \phi\right)$$

where

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

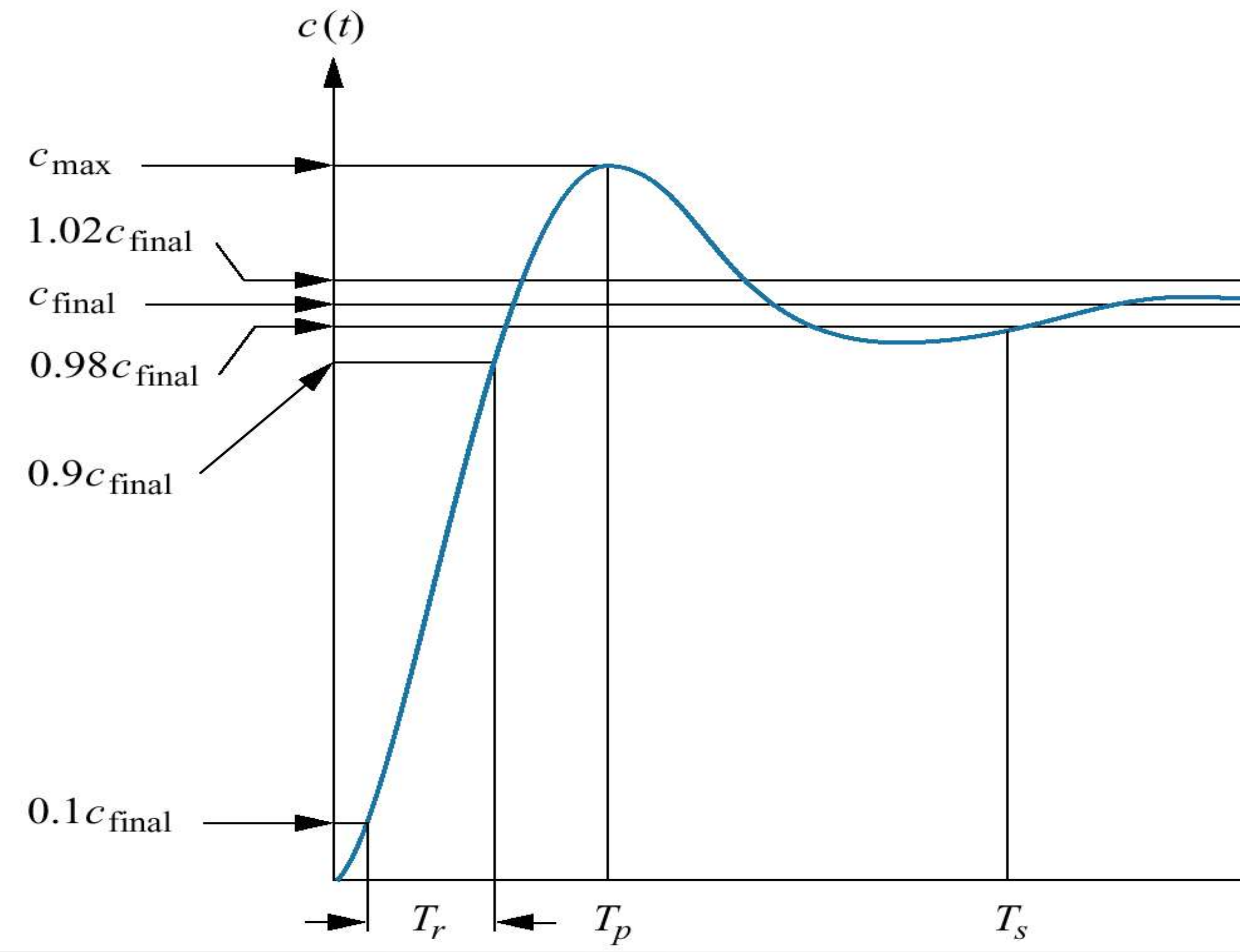
2nd Order System Responce

2nd order underdamped responces for different values of damping ratios



Second Order Systems

Second-order underdamped response specifications



T_r : Rise Time

T_S : Settling Time

T_p : Peak Time

Definitions

For second order systems the specific parameters can be calculated as (details are given in the text book)

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\%O.S. = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$$

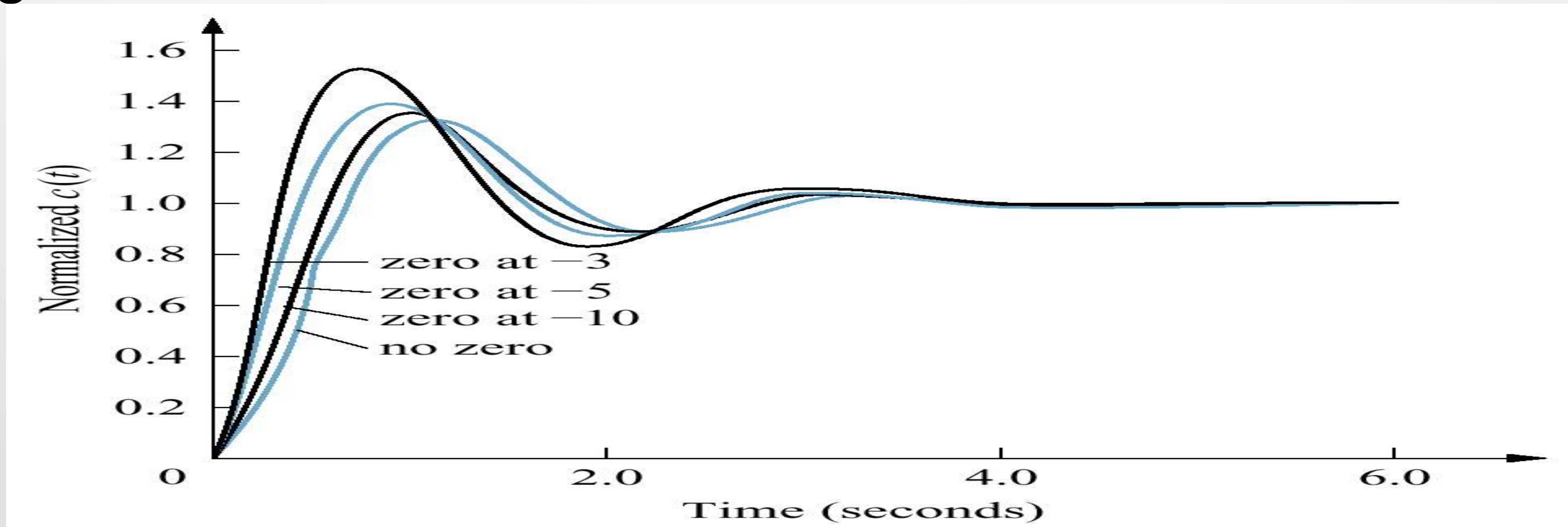
$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

Systems with additional poles

- The previous formulations are valid for systems with one or two poles.
- If a system has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived.
- However, under certain conditions, a system with more than two poles can be approximated as a second-order system that has just two complex dominant poles.
- The certain conditions can be given as that the real parts of those dominant poles are close to zero comparing to the other poles. In other words, the exponential frequency caused by the other poles decays to zero faster.

Additional Zeros ?

- It seems that the zeros of a response affect the residue, or amplitude, of a response component but not the nature of the response—exponential, damped sinusoid, and so on.
- Starting with a two-pole system with poles at (-1≤j2.828), we consecutively add zeros at -3, -5, and -10. The results, normalized to the steady-state value, are plotted in the figure below



Additional Zeros ?

- We can see that the closer the zero is to the dominant poles, the greater its effect on the transient response. As the zero moves away from the dominant poles, the response approaches that of the two-pole system.
- Note that, we added zero in the left half plane only.
  - A zero in the right half plane may effect the response in a different way. The system is said to be nonminimum-phase in this case.
- Assume a three-pole system with a zero. If the pole term, (s+p3), and the zero term, (s+z),cancel out we are left with

$$T(s) = \frac{K\cancel{(s+z)}}{\cancel{(s+p_3)}\,(s^2+as+b)}$$

Stability

<u>Definition :</u>

A linear, time invariant system is said to be stable if its impulse response approaches to zero as time goes to infinity.

The overall behivour of the system is defined by its poles positions.

- When the poles are in the OLHP (open left half plane), that is poles have negative parts system response damps as time goes to infinity (stable system)
- *When poles have positive real parts the response of the system goes to infinity as time approaches to infinity (unstable system)*

Stability

Back to the definition

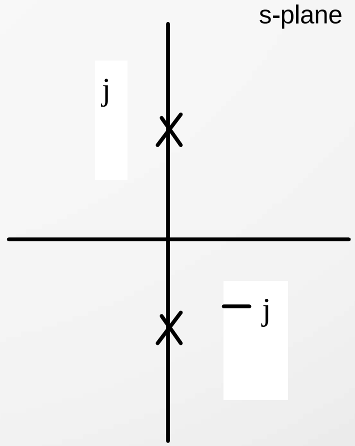
$$\lim_{t \to \infty} h(t) = \mathcal{L}^{-1}\{H(s)\} = 0$$

this simply means *poles* should have negative real parts

Examples

$$H(s) = \frac{1}{s^2 + 1}$$

$$h(t) = \sin(t)$$



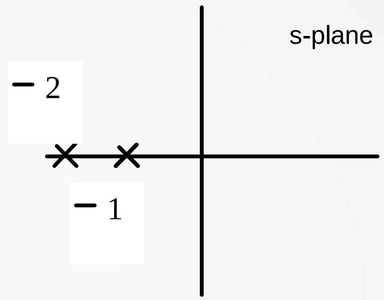
Not Stable

Examples

Example :

$$H(s) = \frac{1}{(s+1)(s+2)}$$

$$h(t) = \exp(-t) - \exp(-2t)$$

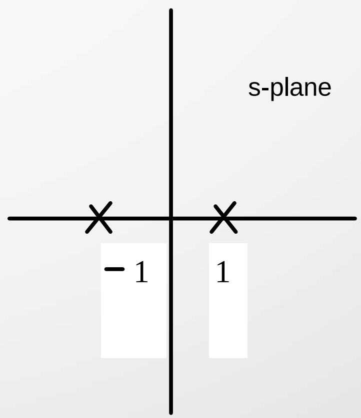


Stable

Example :

$$H(s) = \frac{1}{(s+1)(s-1)}$$

$$h(t) = -\frac{1}{2}\exp(-t) + \frac{1}{2}\exp(t)$$



Not Stable

Stability Test

Do we really need to factor the denominator of the closed-loop (lets say overall for now) transfer function to check the stability of the system ??

Method 1 : Using long division

Method 2: Rought-Hurwirtz Criterian

Stability Check using Long Division

Assume

$$H(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{s^N + \sum_{n=0}^{N-1} a_n s^n}$$

Take the denumarator

$$Q(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

formulate two new polynomials as

$$\mathrm{Q}_1(s) = s^n + a_{n-2}s^{n-2} + \ldots$$

$$Q_2(s) = a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + \ldots$$

Stability Check using Long Division

Form

$$\frac{Q_1(s)}{Q_2(s)} = h_1 s + \cfrac{1}{h_2 s + \cfrac{1}{h_3 s + \cfrac{1}{h_4 s + \dots h_n s}}}$$

If all $h_1, h_2, h_3, ..., h_n$ are positive then the roots of $Q(s)$ are in the open left half plane (OLHP).

The system is then stable!!!

Stability Check using Long Division

Example :

check the stability of the system with the transfer function *H(s)* having denomunator as

$$Q(s) = s^3 + 6s^2 + 12s + 8$$

Solution : Form

$Q_1(s) = s^3 + 12s$ which yields $\frac{Q_1(s)}{Q_2(s)} = \frac{s^3 + 12s}{6s^2 + 8}$

$Q_2(s) = 6s^2 + 8$

$$\frac{Q_1(s)}{Q_2(s)} = \frac{1}{6}s + \frac{1}{\frac{9}{16}s + \frac{1}{\frac{4}{3}s}}$$

$h_1 = 1/6, h_2 = 9/16, h_3 = 4/3$ all positive system is STABLE

Rought-Hurwitz Criterian

Not this Week....

we will came back after

Block Diagram representation of Systems